KINEMATICS OF HIGHER-SPIN FIELDS

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Abstract

This dissertation argues a quantum theory for higher-spin fields. Causality is discussed in both classical and quantum senses in a field theory. Renormalizability is also considered. Equations of motion are derived from a generalised concept of linearised Christoffel symbols for these higher-spin fields. It is seen that a minimally-coupled Rarita-Schwinger field violates causality, and that supergravity restores it at a classical level. Then these fields are quantised with generalised polarisation 4-vectors. Feynman rules are constructed in momentum space.

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CHAPTER I

INTRODUCTION

Quantum field theory (QFT) is a quantum theory in which classical fields such as electromagnetic fields A_μ are quantised into operators in the Hilbert space for the purpose of creating or annihilating particles or antiparticles. Mathematically particles or antiparticles (more correctly the states of particles or antiparticles) are identified with eigenstates of these operators, and since each state of particles or antiparticles belongs to the Hilbert space, every state can be expressed as a linear combination of other states.

QFT is a generalised notion of the relativistic quantum mechanics, where special theory of relativity is incorporated into quantum mechanics, that is, quantum mechanics is reformulated so that it can preserve Lorentz covariance. QFT is also regarded as an extension of quantisation of a many-particle system; some people call this procedure second quantisation. Since Lorentz covariance is inevitably lost[1] for a many-

[1]This is because the speed of transmission of information is finite, and the corresponding purely classical Lagrangian contains only the information of both the positions and the velocities of particles, not that of the internal degrees of freedom of purely classical fields (e.g. a classical electromagnetic field). Therefore, a Darwin Lagrangian is introduced for such a

particle system[1] where each particle interacts with each other, especially in condensed matter theory, the classical Lagrangian formulation should not work[2] in a precise manner. Thus a relativistic situation of such a system requires that the particles themselves should be treated as a (Lorentz covariant) continuous object in which particles are grouped together. The continuous object is called a field. In other words, a many-particle system turns into a field in a relativistic case. This is a pedagogical introduction of the notion of a field in condensed matter theory.

From the standpoint of particle physics, the fact that the relativistic quantum mechanics faces two setbacks[3] motivates one to introduce the notion of a field in quantum mechanics. Once a Klein-Gordon equation is substituted for a Schrödinger equation, the appearance of a negative energy may break down the notion of potential energy in physics. The second setback is that one is forced to abandon the concept of a particle probability density $\rho = |\phi|^2$ in order to preserve the Lorentz covariance. Thus, in a relativistic situation, the equations of motion for the particle should turn into those for another object which produces or destructs the particle. This object is called a field. The notion of the negative energy is solved with the introduction of an antiparticle.

In condensed matter theory a field is regarded as an assembly of particles, while in particle physics a field is interpreted as 'machinery' which create or annihilate particles or antiparticles. Both approaches are same essentially in that particles (or antiparticles) are created or annihilated in vacuum.

QFT treats spin-0, spin-1/2 and spin-1 fields, and such lower-spin fields are real except for a Higgs field, a fundamental scalar field. Even though QFT can treat a spin-3/2 field, it was known[4] that the field violated causality once it coupled with a particular external field. It was pointed out[5] that this fact implied either that there was a defect in Lagrangian approach or that there were no fundamental particles with spin-(s+1/2) ($s \geq 1$) . Indeed, even though supersymmetry (SUSY) predicts the existence of a graviton, which has spin-2, and its superpartner, a gravitino, which is a fundamental particle with spin-3/2 , such particles have not been observed so far. They are still phenomenological particles.

In the second chapter, causality for a field theory is reviewed in both classical and quantum senses, and renormalizability is also considered. In the third chapter higher-spin fields are discussed. This dissertation is closed with the conclusion.

system as an approximation.

Throughout this dissertation, the following notation

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1, -1)\ , \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{0.0.1}$$

$$\int d^4 e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k) \tag{0.0.2}$$

is used in d-dimensional space.

CHAPTER II

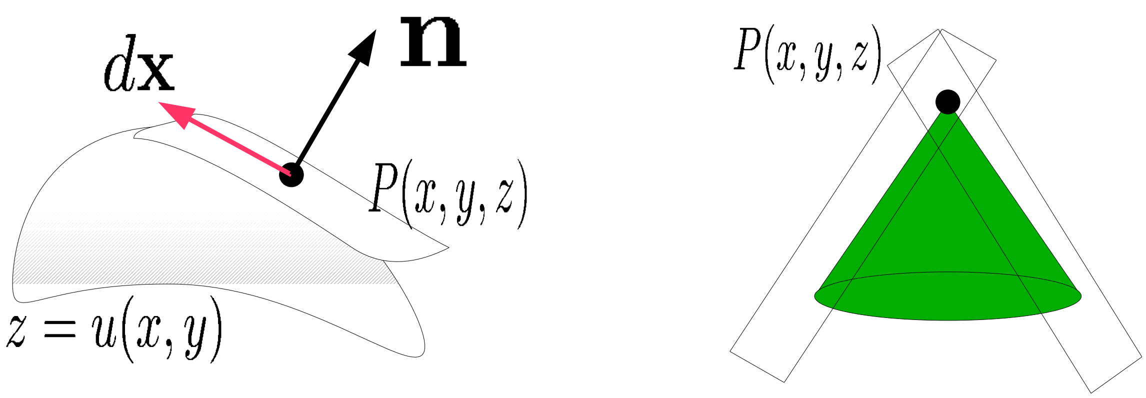
CAUSALITY AND RENORMALIZABILITY

The present chapter reviews causality and renormalizability in the field theory. In the first section, causality is discussed in a classical sense, and the relevant concepts such as a characteristic determinant and hyperbolicity are given. It is seen that the existence of a timelike normal vector to a characteristic surface of a solution for a partial differential equation makes the solution acausal in a classical level. Then it turns out that in quantum mechanics, equal-time commutation relations between a particle sector and its antiparticle sector play an important role in judging the causality of a quantum field. The section owes most of relevant definitions and ideas to Courant and Hilbert[6] in a classical level, and relies on Tong's website[7] in a quantum level.

This chapter is closed with a review of renormalizability. The power-counting method is considered. It turns out that dimension of a coupling constant tells us the renormalizability of the theory. Then we see that counter terms are used in renormalized perturbation theory, where renormalization conditions are taken. One-loop structure of ϕ^4 theory is also studied. The book[8] written by Peskin and Schroeder is referenced for this chapter.

1 Causal behaviour of fields

Quantum field theory is based on special theory of relativity, and so any information cannot travel faster than the speed of light. In this section we discuss, in both quantum and classical sense, causal behaviour of lower-spin fields, explaining mechanism of causality. Firstly we set basic definitions of a characteristic surface, a characteristic determinant and hyperbolicity. Then we see that the existence of a timelike



(a) An integral surface $z = u(x,y)$ and its tangent plane with the normal vector $\mathbf{n}$at a point $P(x,y,z)$.

(b) A Monge cone with its vertex at a point $P(x,y,z)$, which touches the planes of the one-parameter family.

Figure 1: An integral surface and a Monge cone

normal vector to the characteristic surface makes it possible that a solution of the PDE behaves in an acausal way. It turns out that this leads to a criterion for judging whether a field violates causality or not. It is also seen that classically Dirac equations and Maxwell equations are hyperbolic. Then, considering equal-time commutation relations, we realise that quantised Klein-Gordon equations have causal behaviour in a quantum sense. It is seen that if a particle moves in an acausal way, its antiparticle cancels it out. This is the reason why antiparticles exist in QFT.

1.1 Hyperbolic systems

In this subsection, for PDEs, a definition of hyperbolicity and characteristic surface are given.

Geometric interpretation of a PDE (first order): To begin with, let us consider a first order PDE which consists of two variables x, y .

$$A(x,y,z)\frac{\partial u}{\partial x} + B(x,y,z)\frac{\partial u}{\partial y} = C(x,y,z) \tag{1.1.1}$$

$$A^2 + B^2 \neq 0 \tag{1.1.2}$$

If only two variables, e.g. x, y , are used in a partial differential equation, then one solution $z = u(x,y)$ of the PDE is interpreted as a surface; this surface is called an *integral surface* in the xyz-space. In (1.1.1)

, the integral surface $z = u(x,y)$ should have a tangent plane, at a point $P(x,y,z)$. The normal vector $\mathbf{n}$ of the tangent plane is written as

$$\mathbf{n} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ -1 \end{pmatrix} \tag{1.1.3}$$

which is related with the differential equation $A\dfrac{\partial u}{\partial x} + B\dfrac{\partial u}{\partial y} = C$ (See Figure 1a). This equation indicates that the tangent planes of all integral surfaces[2], passing through the point $P(x,y,z)$, belong to a set. This set is interpreted as a straight line or a pencil which has the axis satisfying the relations:

$$d\mathbf{x} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}\Bigg|_P , \qquad \text{and } d\mathbf{x}\cdot\mathbf{n} = 0 \tag{1.1.4}$$

at the point $P(x,y,z)$. We call this pencil and axis Monge pencil and Monge axis, respectively. A line characteristic element is formed by the point $P(x,y,z)$ and the direction of Monge axis passing through P . The *characteristic curves* of the PDE are described by (1.1.4) , and if a parameter λ along the characteristic curves, then the differential equations reduce

$$d\mathbf{x} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} d\lambda \quad . \tag{1.1.5}$$

Finding surfaces which are, at every point, tangent to the Monge axis corresponds to integrating the PDE. Therefore, we can say that an integral surface of the PDE is a surface $u(x,y)$ generated by a one-parameter family of characteristic curves. Indeed, if we consider a one-parameter family $\{C_i\}$ of curves defined by

$$\frac{dx}{d\lambda} = A\,, \quad \frac{dy}{d\lambda} = B \quad , \tag{1.1.6}$$

then inevitably we have

$$\frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{d\lambda} \tag{1.1.7}$$

$$(1.1.1) \to = A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = C \quad , \tag{1.1.8}$$

[2] We may say that an integral surface of the PDE is the same as a solution of the PDE.

and so the one-parameter family satisfies (1.1.5) , which means that the one-parameter family is made up of characteristic curves. For non-linear PDEs, although tangent planes do not constitute a pencil of planes through a line, they make a one-parameter family which envelope a conical surface with $P(x,y,z)$ as a vertex. This cone is called a Monge cone (See Figure 1b). For example, we have a general PDE

$$G(x,y,z,p,q)=0\ , \quad \text{where } p=\frac{\partial z}{\partial x}\ ,\ \ q=\frac{\partial z}{\partial y}\ ,\ \ z=u(x,y) \quad , \tag{1.1.9}$$

and

$$\left(\frac{\partial G}{\partial p}\right)^2+\left(\frac{\partial G}{\partial q}\right)^2\neq 0 \tag{1.1.10}$$

is required. The differential equation assigns a Monge cone to each point $P(x,y,z)$ in the space. Alternatively, a Monge cone can be expressed as

$$\frac{dx}{dy}=\frac{\frac{\partial G}{\partial p}}{\frac{\partial G}{\partial q}}\ ,\quad \frac{dy}{du}=\frac{\frac{\partial G}{\partial q}}{p\frac{\partial G}{\partial p}+q\frac{\partial G}{\partial q}}\ ,\quad \frac{dx}{du}=\frac{\frac{\partial G}{\partial p}}{p\frac{\partial G}{\partial p}+q\frac{\partial G}{\partial q}} \tag{1.1.11}$$

where $p=p(\lambda),\ q=q(\lambda)$, and this relations[3] are thought to be the representation of the Monge cone dual that obtained by (1.1.10) . The directions of the generators of the Monge cone are called characteristic directions. For a quasi-linear PDE's case, only one characteristic direction belongs to each point in space. We call space curves having a characteristic direction at each point Monge curves[4]. For the Monge curves, an appropriate parameter s is used, and (1.1.11) are expressed as

$$\frac{dx}{ds}=\frac{\partial G}{\partial p}\ ,\quad \frac{dy}{ds}=\frac{\partial G}{\partial q}\ ,\quad \frac{du}{ds}=p\frac{\partial G}{\partial p}+q\frac{\partial G}{\partial q} \quad ; \tag{1.1.12}$$

these three conditions are called the strip condition. Simultaneously both a space curve and its tangent plane are defined at each point by the functions $x(s), y(s), u(s), p(s), q(s)$. Let us call a configuration made up of a curve and a family of a tangent planes to this curve a strip.

Several types of PDE: Discussion may become easier by using a differential operator $L[u]$. Let us consider a linear differential operator of second order

$$L[u]=A\frac{\partial^2 u}{\partial x^2}+2B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2} \quad , \tag{1.1.13}$$

[3]These relations (1.1.11) are derived by three relations: $\frac{du}{d\sigma}=p\frac{dx}{d\sigma}+q\frac{dy}{d\sigma}$, $\frac{dp}{d\lambda}\frac{dx}{d\sigma}+\frac{dq}{d\lambda}\frac{dy}{d\sigma}=0$ and $\frac{\partial G}{\partial p}\frac{dp}{d\lambda}+\frac{\partial G}{\partial q}\frac{dq}{d\lambda}=0$, where σ means the distance from the vertex of the cone. Here x,y,u are regarded as functions of σ along a fixed generator. The last equation is obtained by differentiating the both sides of $G=0$ with regard to λ .

[4]They are also called focal curves.

and we construct a more general PDE which is not necessarily linear:

$$L[u] + h(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \quad , \tag{1.1.14}$$

where $h(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ is a function. Then, introducing new independent variables:

$$\xi = \Phi(x, y) \ , \quad \eta = \Psi(x, y) \quad , \tag{1.1.15}$$

we alter the PDE (1.1.14) into a simple normal form.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \Phi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \Psi}{\partial x} \ , \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \Phi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \Psi}{\partial y} \tag{1.1.16}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \Psi}{\partial x}\right)^2 \tag{1.1.17}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y} \tag{1.1.18}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \Phi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \Psi}{\partial y}\right)^2 \tag{1.1.19}$$

The differential operator $L[u]$ may be transformed into

$$T[u] = \Gamma_1 \frac{\partial^2 u}{\partial \xi^2} + 2\Gamma_2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \Gamma_3 \frac{\partial^2 u}{\partial \eta^2} \quad , \tag{1.1.20}$$

where

$$\Gamma_1 = A\left(\frac{\partial \Phi}{\partial x}\right)^2 + 2B\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + C\left(\frac{\partial \Phi}{\partial y}\right)^2 \tag{1.1.21}$$

$$\Gamma_2 = A\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial x} + B\left(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial x}\right) + C\frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial y} \tag{1.1.22}$$

$$\Gamma_3 = A\left(\frac{\partial \Psi}{\partial x}\right)^2 + 2B\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y} + C\left(\frac{\partial \Psi}{\partial y}\right)^2 \quad . \tag{1.1.23}$$

We see that there are relations between (A, B, C) and $(\Gamma_1, \Gamma_2, \Gamma_3)$:

$$\Gamma_1\Gamma_3 - (\Gamma_2)^2 = (AC - B^2)(\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial x})^2 \tag{1.1.24}$$

$$Q(l, m) = Al^2 + 2Blm + Cm^2 = \Gamma_1\lambda^2 + 2\Gamma_2\lambda\mu + \Gamma_3\mu^2 \quad , \tag{1.1.25}$$

where (l, m) and (λ, μ) are related as

$$l = \lambda\frac{\partial \Phi}{\partial x} + \mu\frac{\partial \Phi}{\partial y} \ , \quad m = \lambda\frac{\partial \Psi}{\partial x} + \mu\frac{\partial \Psi}{\partial y} \quad . \tag{1.1.26}$$

Next we impose conditions:

$$\text{case 1} \quad \Gamma_1 = \Gamma_3 \ , \quad \Gamma_2 = 0 \tag{1.1.27}$$

$$\text{case 2} \quad \Gamma_1 = -\Gamma_3 \ , \quad \Gamma_2 = 0 \quad \text{or} \Gamma_1 = \Gamma_3 = 0 \tag{1.1.28}$$

$$\text{case 3} \quad \Gamma_2 = \Gamma_3 = 0 \quad . \tag{1.1.29}$$

Correspondingly, for $Q(l,m) = 1$, and for fixed point (x,y) the differential operator $T[u]$ are called

$$\text{case 1. elliptic if} \quad AC - B^2 > 0 \tag{1.1.30}$$

$$\text{case 2. hyperbolic if} \quad AC - B^2 < 0 \tag{1.1.31}$$

$$\text{case 3. parabolic if} \quad AC - B^2 = 0 \quad , \tag{1.1.32}$$

and the differential operator takes such forms:

$$\text{case 1.} \quad T[u] = \Gamma_1 \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \right) + (\text{lower order terms}) \tag{1.1.33}$$

$$\text{case 2.} \quad \begin{cases} T[u] = \Gamma_1 \left(\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} \right) + (\text{lower order terms}) \\ \text{or} \\ T[u] = 2\Gamma_2 \frac{\partial^2 u}{\partial \xi \partial \eta} + (\text{lower order terms}) \end{cases} \tag{1.1.34}$$

$$\text{case 3.} \quad T[u] = \Gamma_1 \frac{\partial^2 u}{\partial \xi^2} + (\text{lower order terms}) \quad , \tag{1.1.35}$$

and additionally the normal forms of the differential equation are

$$\text{case 1.} \quad \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + (\text{lower order terms }) = 0 \tag{1.1.36}$$

$$\text{case 2.} \quad \begin{cases} \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + (\text{lower order terms }) = 0 \\ \text{or} \\ \frac{\partial^2 u}{\partial \xi \partial \eta} + (\text{lower order terms }) = 0 \end{cases} \tag{1.1.37}$$

$$\text{case 3.} \quad \frac{\partial^2 u}{\partial \xi^2} + (\text{lower order terms }) = 0 \quad . \tag{1.1.38}$$

Characteristic curves and determinants: Now we consider a system of k equations for a function vector $u(x,y) = (u_1, \ldots u_k)$ in 2 independent variables x, y . Its differential operator is

$$L_j[u] = A_{ij} \frac{\partial u_i}{\partial x} + B_{ij} \frac{\partial u_i}{\partial x} + D_j \ , \quad j = 1, \ldots, k \ , \tag{1.1.39}$$

where[5] $A = (A_{ij}),\ B = (B_{ij})$ are[6] k by k matrices. If we express (1.1.39) as a matrix form, we have

$$L[u] = A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + D \quad , \tag{1.1.40}$$

where L, D and u stand for vectors. Now, considering $L[u] = 0$, we confront the Cauchy initial value problem, that is, provided that initial values of the vector u on a curve C: $\phi(x, y) = 0$ with $\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \neq 0$, the first derivatives $\frac{\partial u}{\partial x_i}$ on C so that $L[u] = 0$ is satisfied on the strip. On C, the interior[7] derivative $\frac{\partial u}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial\phi}{\partial y}$ is known, and there is a relation such that

$$\frac{\partial u}{\partial y} = -\tau\frac{\partial u}{\partial x} + \cdots \quad , \quad \tau = -\frac{\frac{\partial\phi}{\partial y}}{\frac{\partial\phi}{\partial x}} \tag{1.1.41}$$

where the dots refer to quantities known on C. Using this in (1.1.39) , we have

$$L_j[u] = (A_{ij} - \tau B_{ij})\frac{\partial u_i}{\partial x} + \cdots = 0 \ , \quad j = 1, \ldots, k \tag{1.1.42}$$

that is, a system of linear equations for the k derivatives $\frac{\partial u_i}{\partial x}$. Thus a necessary and sufficient condition for determining all the derivatives along C is

$$Q = \det(A - \tau B) \neq 0 \quad , \tag{1.1.43}$$

and Q is called the characteristic determinant of the system (1.1.39) . If Q does not vanish along the curves $\phi = \text{const}$, then such curves are called free. For these curves, continuation of initial values into a 'strip' satisfying (1.1.39) is possible, and we can choose initial values arbitrarily. If the algebraic equation $Q = 0$ of order k has a real solution $\tau(x, y)$, then the curves C , which are defined by the ordinary differential equation

$$\frac{dx}{dy} = \tau \ , \quad \text{or} \quad Q(x, y, \frac{dx}{dy}) = 0 \quad , \tag{1.1.44}$$

are called characteristic[8] curves. If there are no real solutions τ for the equation $Q = 0$, then all curves are free, that is, their initial values can be always continued into a strip uniquely. Then the system is

[5] Let us emphasise that here the summation convention is taken for the indices which appear twice in a single term.

[6] We assume that either A or B is a non-singular matrix.

[7] For a function $f(x, y)$ in a curve $\phi(x, y) = 0$ with $\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \neq 0$, a differentiation $\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}$ is said to be interior if $\alpha\frac{\partial\phi}{\partial x} + \beta\frac{\partial\phi}{\partial y} = 0$. In particular $\frac{\partial\phi}{\partial y}\frac{\partial f}{\partial x} - \frac{\partial\phi}{\partial x}\frac{\partial f}{\partial y}$ is an interior differentiation of f .

[8] Generally, it is not possible that initial values are continued into an integral strip for characteristic curves.

called elliptic. By contrast, if the equation $Q = 0$ possesses k real solutions which are distinct each other, the system is called totally hyperbolic.

If τ is a real solution of (1.1.43) , along C it is possible to solve the system of linear homogeneous equations for the vector l with components $l_1, \ldots, l_k$:

$$l_j(A_{ij} - \tau B_{ij}) = 0\ , \quad \text{or} \quad l(A - \tau B) = 0 \quad , \tag{1.1.45}$$

and $l_j L[u] = lL[u]$ of the differential equations (1.1.39) can be expressed as the characteristic normal form:

$$l_j L_j[u] = l_j B_{ij}(\frac{\partial u_i}{\partial y} + \tau \frac{\partial u_i}{\partial x}) + \cdots = 0 \tag{1.1.46}$$

or

$$lL[u] = lB(\frac{\partial u}{\partial y} + \tau \frac{\partial u}{\partial x}) + \cdots = 0 \quad , \tag{1.1.47}$$

where all the unknowns are differentiated along the characteristic curve corresponding to τ . Therefore, in the hyperbolic case, where k such families of characteristic curves exist, the system is replaced by equivalent one in which equation has differentiation only in one, characteristic, direction.

Generalisation to n independent variables: So far we restricted variables within 2. Now we consider a system of first order with n independent[9] variables $x = (x_1, \ldots, x_n)$. In this case,

$$L_j[u] = A_{ij,\nu} \frac{\partial u_i}{\partial x_\nu} + B_j = 0\ , \quad j = 1, \ldots, k \tag{1.1.48}$$

$$\nu = 1, \ldots, n \quad , \tag{1.1.49}$$

where[10] $A_{ij,\nu}$ and B_j depend on x and possibly also on u . In matrix notation,

$$L[u] = A_\nu \frac{\partial u}{\partial x_\nu} + B = 0 \quad , \tag{1.1.50}$$

where A_ν are k by k matrices, and B is a vector.

Then a surface $C : \phi(x) = 0$ with $\nabla\phi = \text{grad}\phi = 0$ is considered. On the surface C, the quantity

$$A = A_\nu \frac{\partial \phi}{\partial x_\nu} \tag{1.1.51}$$

[9]In this case, in a manifold $\phi(x_1, \ldots, x_n) = 0$ with $\nabla\phi \neq 0$, the interior differentiation of a function $f(x_1, \ldots, x_n)$ of n variables $x_1, \ldots, x_n$ is $c_\nu \frac{\partial f}{\partial x_\nu}$, or a linear combination of $c_1 \frac{\partial f}{\partial x_1}, \ldots, c_n \frac{\partial f}{\partial x_n}$, provided that $c_\nu \frac{\partial \phi}{\partial x_\nu} = 0$ is satisfied.

[10]The summation convention is taken with regard to ν, i .

is called the characteristic matrix, and the characteristic determinant[11] is

$$Q(\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_n}) = \det A \quad , \tag{1.1.52}$$

in this case. We can set initial values of a vector u on C. Let us draw attention to the fact that in C $\frac{\partial u}{\partial x_\nu}\frac{\partial\phi}{\partial x_n} - \frac{\partial u}{\partial x_n}\frac{\partial\phi}{\partial x_\nu}$ is an interior derivative of u. Thus, assuming that $\frac{\partial\phi}{\partial x_n} = 0$, in C $\frac{\partial u}{\partial x_\nu}$ is known from the data if only the one outgoing derivative $\frac{\partial u}{\partial x_n}$ is known. Multiplying (1.1.50) by $\frac{\partial\phi}{\partial x_n}$, we see that

$$\frac{\partial\phi}{\partial x_n}L[u] = A_\nu\frac{\partial u}{\partial x_n}\frac{\partial\phi}{\partial x_\nu} + \frac{\partial\phi}{\partial x_n}B = A\frac{\partial u}{\partial x_n} + \mathcal{J} = 0 \quad , \tag{1.1.53}$$

where $\mathcal{J} = \frac{\partial\phi}{\partial x_n}B$ can be an interior differential operator on u in C. Therefore, provided that the characteristic determinant Q does not vanish, $\frac{\partial u}{\partial x_n}$ is uniquely determined by the system (1.1.53) of linear differential equations for the vector $\frac{\partial u}{\partial x_n}$, and in this case the surface C is called free.

On the other hand, if the characteristic determinant is zero, then a null vector l exists such that $lA = 0$. Multiplying (1.1.53) by l gives rise to

$$l\frac{\partial\phi}{\partial x_n}L[u] = l\mathcal{J} = 0 \quad , \tag{1.1.54}$$

which is, along C, expressed by an interior differential operator on the data, and this operator $l\mathcal{J}$ does not have $\frac{\partial u}{\partial x_n}$. This suggests that $l\mathcal{J}$ is a differential relation which restricts the initial values of u on C. If the characteristic determinant vanishes along C, the surface is called a characteristic surface. Then there exists a characteristic linear combination

$$lL[u] = l_jL_j[u] = \Lambda[u] \tag{1.1.55}$$

of the differential parameters L_j such that in Λ the differentiation of the vector u on C is interior, and a relation among the initial data are described by the equation $\Lambda[u] = 0$. Thus we cannot take these data arbitrarily.

Next let us categorise the partial differential equations in n independent variables. If one cannot realise the homogeneous algebraic equation $Q = 0$ in the quantities $\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_n}$ by any real set of values (except $\frac{\partial\phi}{\partial x_\nu} = 0$) , then there exist no characteristics, and the system is called elliptic. By contrast, if the equation $Q = 0$ has k distinct real solutions $\frac{\partial\phi}{\partial x_n}$ for arbitrarily prescribed values of $\frac{\partial\phi}{\partial x_1},\ldots,\frac{\partial\phi}{\partial x_{n-1}}$ (or if a corresponding statement is true after a suitable coordinate transformation), then we call the system totally hyperbolic.

[11] It is also called the characteristic form.

Hyperbolicity for higher order PDEs: The cases of higher-order PDEs should be discussed for hyperbolicity. Here we denote $\frac{\partial}{\partial x_\nu}$ by D_ν for convenience. In cases of higher order,

$$L[u] = H(D_1, \dots, D_n)u + K(D_1, \dots, D_n)u + f(x) = 0 \quad , \tag{1.1.56}$$

where H is a homogeneous polynomial in D of degree m and K is a polynomial of degree lower than m, assuming that all the coefficients are continuous functions of x. Let us define the Cauchy data as the given initial values. Provided that $\frac{\partial \phi}{\partial x_n} \neq 0$ on the surface $C : \phi(x_1, \dots, x_n) = 0$, the Cauchy data is made up of the values of the function u and its first $m-1$ derivatives on the surface C.

Let us introduce new coordinates as independent variables. One chooses ϕ as one of these coordinates, and $\lambda_1, \dots, \lambda_{n-1}$ as interior coordinates in the surfaces $\phi = const$. Then one can write all the m-th derivatives of a function u as combinations of the m-th 'outgoing' derivative $\frac{\partial^m u}{\partial \phi^m}$ with terms which possess at most (m - 1)-fold differentiation with respect to ϕ , and therefore all the m-th derivatives of a function u are known from the data. The equation is

$$Q(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n})\frac{\partial^m u}{\partial \phi^m} + \dots = 0 \quad , \tag{1.1.57}$$

where the dots stand for terms which are known on C from the data. This equation for u has a unique solution if and only if $Q \neq 0$. If the characteristic determinant vanishes on C, an internal condition for the data is represented by the equation.

Hence, in order to determine the condition under which arbitrary data on C determine uniquely the m-th derivatives of u on C, it is necessary and sufficient that the characteristic determinant

$$Q(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n}) = H(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n}) \tag{1.1.58}$$

is not zero on C. If the surface C is characteristic, that is, the surface satisfying $Q = 0$, then $Hu + Ku$ is an internal differential operator of order m on C. This implies that m-th derivatives are contained in the differential operator only in such a way that they combine into internal first derivatives of operators of order m-1 , and therefore they are known on C from the data.

1.2 Mathematics for classical causality

This subsection argues causality in a classical sense from the view of mathematics. The normal vector and a tangent vector to a characteristic surface are considered, and they are orthogonal. It is seen that

(a) A hypersurface S and its normal vector ξ_μ at a point p

(b) A characteristic surface C and its normal vector $\xi_\mu = \partial_\mu \phi$. An intuitive explanation why ξ_μ is normal to C. The covectors dx^ν are chosen as they are tangent to the characteristic surface.

Figure 2: Hypersurfaces S, C and their corresponding normal vectors

if the normal vector is timelike, a tangent vector is spacelike, which states that the characteristic surface has acausal properties. By searching for the existence of a timelike normal vector to the characteristic surface, it may be possible to know whether a solution of a PDE violates causality or not.

Normal vectors to hypersurfaces: Let $\mathcal{M}$ be an m-dimensional manifold, and S be its hypersurface, that is, an $(m-1)$-dimensional embedded submanifold of $\mathcal{M}$. We denote the tangent spaces, at a point p $(\in \mathcal{M})$, of $\mathcal{M}$ and S by $T_p\mathcal{M}$ and T_pS . We starts our discussion by considering an orthogonal vector to the hypersurface S . Since $T_pS \subset T_pM$, there exists a vector $\xi_\mu \in T_pM$ such that

$$g_{\mu\nu}\xi^\mu v^\nu = 0 \tag{1.2.1}$$

for $\forall v_\mu \in T_pS$. Then we say that the vector ξ_μ is normal to the hypersurface S (See Figure 2a). If the metric is pseudo-Riemannian, it is possible that the normal vector is a null vector:

$$g_{\mu\nu}\xi^\mu\xi^\nu = 0 \quad , \tag{1.2.2}$$

and in this case as far as (1.2.2) holds true, the normal vector belongs[12] to the tangent space of S , that is $\xi_\mu \in T_p S$, and S is called a null hypersurface.

In the previous section, we see a characteristic surface in an n-dimensional manifold $\mathcal{N}$, and it may be possible to regard a characteristic surface as an assembly of characteristic curves. Now we find that a characteristic surface $C : \phi(x^1, \dots x^m) = const$ has its normal vector $\xi_\mu \in T_p N$ at a point p. It is defined as

$$\xi_\mu \equiv \partial_\mu \phi \quad . \tag{1.2.3}$$

Why is it normal to the hypersurface C ? Intuitively,

$$0 = d\phi = \frac{\partial \phi}{\partial x^\nu} dx^\nu \tag{1.2.4}$$

$$= \xi_\nu dx^\nu \quad , \tag{1.2.5}$$

and the covectors dx^ν are taken as they are tangent to the characteristic surface. Therefore ξ_μ is normal to the characteristic surface C (See Figure 2b).

Rigorously, we use a corollary of the Frobenius theorem. The corollary states that the necessary and sufficient condition that a vector field ξ_μ should be hypersurface orthogonal is

$$\xi_{[\kappa} \nabla_\lambda \xi_{\mu]} = 0 \quad , \tag{1.2.6}$$

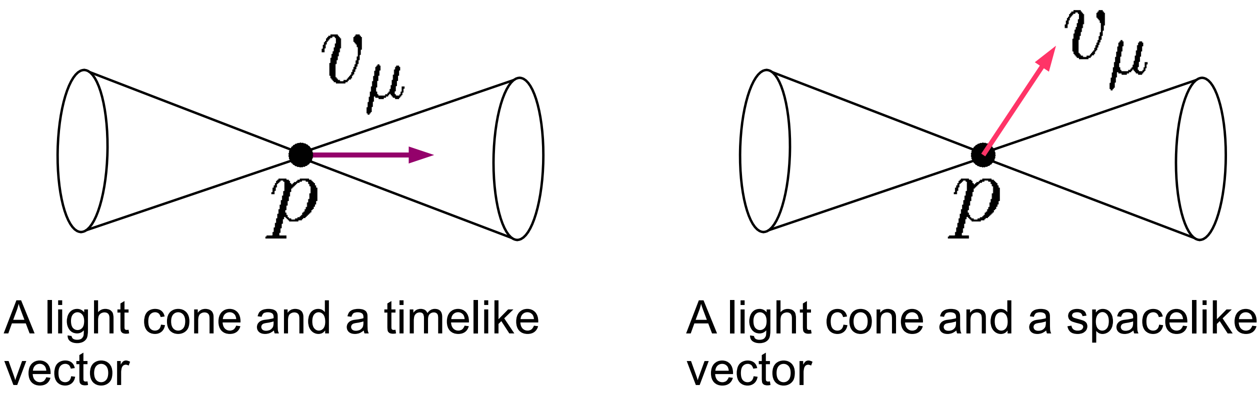
where the nabla stands for the covariant derivative operator in a pseudo-Riemannian space. In a Minkowski space, we find that the normal vector $\xi_\mu = \partial_\mu \phi$ satisfies the condition (1.2.6) .

Local causality and timelike vectors: We know that a curve is causal locally, at a point p, in the manifold $\mathcal{M}$ if its tangent vectors v_μ are timelike or null at that point, that is,

$$g_{\mu\nu} v^\mu v^\nu > 0 \ , \quad \text{or } g_{\mu\nu} v^\mu v^\nu = 0 \quad , \tag{1.2.7}$$

respectively. The curve locally[13] runs inside the light cone whose centre is p if v_μ is timelike, passing through the point p. The curve locally runs along the surface of the cone if the tangent vector is null. If the vector is spacelike, ie $g_{\mu\nu} v^\mu v^\nu < 0$, then the curve locally goes outside the cone (See Figure 3). One

[12]By contract, if the metric is Riemannian, then $\xi_\mu \notin T_p S$.
[13]In this context, locally means an arbitrarily small region near p.



(a) A light cone and a timelike vector v_μ , $(v_\mu v^\mu > 0)$. (b) A light cone and a spacelike vector v_μ , $(v_\mu v^\mu < 0)$.

Figure 3: Light cones and tangent vectors. At the point p, the parameter λ takes zero.

realises this fact by considering a curve $C : x^\mu(\lambda) = (c\lambda, \mathbf{x}(\lambda))$ which is parametrised by one parameter λ :

$$dx^\mu = \frac{dx^\mu}{d\lambda} d\lambda = v^\mu d\lambda \tag{1.2.8}$$

$$\text{and so} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} v^\mu v^\nu d\lambda d\lambda \quad , \tag{1.2.9}$$

where $v^\mu = \dfrac{dx^\mu}{d\lambda} = (c, \mathbf{u})$. Indeed, if the vector is spacelike, then $v_\mu v^\mu = c^2 - \mathbf{u}^2 < 0$, that is, the speed of the object exceeds that of light.

Now let us how classical causality is linked to a characteristic surface C and its normal vectors ξ_μ . This is the essential part of this subsection.

Theorem 1.1 *Let $\xi_\mu \in T_pM$, $v_\mu \in T_pC$. The characteristic surface C is a hypersurface of the manifold $\mathcal{M}$. Suppose ξ_μ is timelike at the point p, and is normal to the characteristic surface C , that is $\xi_\mu v^\mu = 0$.*

Then any tangent vector v_μ of C is spacelike, and accordingly the characteristic surface is spacelike, that is, ca (1.2.10)

Using this theorem, we can investigate whether a solution of equations of motion for fields is spacelike or not.

We finalise this subsection by making a proof of the theorem. In a Minkowski space $(\mathbf{R}^{1,3},\ \eta_{\mu\nu})$, we take the normal vector ξ_μ ($\in T_pM$) and an arbitrary tangent vector v_μ ($\in T_pC$) to the characteristic surface C as

$$\xi_\mu = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad , \quad v_\mu = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \quad . \tag{1.2.11}$$

The assumption says that

$$\xi_\mu v^\mu = aA - bB - cC - dD = 0 \tag{1.2.12}$$

$$\Leftrightarrow aA = bB + cC + dD \quad . \tag{1.2.13}$$

Since ξ_μ is timelike,

$$\xi_\mu \xi^\mu = a^2 - b^2 - c^2 - d^2 > 0 \tag{1.2.14}$$

$$\text{and immediately } -(c^2 + d^2) > b^2 - a^2 \tag{1.2.15}$$

$$-(b^2 + d^2) > c^2 - a^2 \ , \quad -(b^2 + c^2) > d^2 - a^2 \tag{1.2.16}$$

are obtained. Then, by using (1.2.13) , we consider

$$a^2 \left(A^2 - B^2 - C^2 - D^2\right) = (b^2 - a^2)B^2 + (c^2 - a^2)C^2 + (d^2 - a^2)D^2 + 2bcBC + 2cdCD + 2dbDB \quad . \tag{1.2.17}$$

(1.2.15) and (1.2.16) imply that

$$a^2 \left(A^2 - B^2 - C^2 - D^2\right) < -(c^2 + d^2)B^2 - (b^2 + d^2)C^2 - (c^2 + b^2)D^2 + 2bcBC + 2cdCD + 2dbDB \tag{1.2.18}$$

$$= -(cB - bC)^2 - (dB - bD)^2 - (dC - cD)^2 < 0 \quad , \tag{1.2.19}$$

and therefore we have

$$A^2 - B^2 - C^2 - D^2 = v_\mu v^\mu < 0 \quad , \tag{1.2.20}$$

that is, v_μ is spacelike Q.E.D.

1.3 Classical causality for Maxwell and Dirac fields

As examples of hyperbolic PDEs, let us take the Maxwell equations and Dirac equations. First a Maxwell field in vacuum is considered. If we take the Lorentz gauge, the Maxwell equations reduce to

$$\partial_\mu \partial^\mu A^\nu = \partial^2 A^\nu = 0 \quad , \tag{1.3.1}$$

$$\text{which means} \quad \begin{pmatrix} \partial^2 & 0 & 0 & 0 \\ 0 & \partial^2 & 0 & 0 \\ 0 & 0 & \partial^2 & 0 \\ 0 & 0 & 0 & \partial^2 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = 0 \quad . \tag{1.3.2}$$

The characteristic determinant is

$$\begin{aligned} Q(\xi_\mu) &= Q(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \\ &= \left\{ \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 \right\}^4 && (1.3.3) \\ &= (\xi_\mu \xi^\mu)^4 \quad , && (1.3.4) \end{aligned}$$

and so we see that Maxwell equations in the Lorentz gauge are hyperbolic. Since the right-hand side of (1.3.4) is positive for any normal vector except for null, there is no spacelike characteristic surface. We find that a classical Maxwell field with the Lorentz gauge is causal.

Hyperbolicity of Dirac equations: Next we see the hyperbolicity of the Dirac equations.

$$(i\gamma^\mu \partial_\mu - mc)\psi = 0 \tag{1.3.5}$$

Taking two 2-by-2 matrices M_1, M_2 and M_3 as

$$M_1 = i \begin{pmatrix} -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \quad , \quad M_2 = i \begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix} \quad , \tag{1.3.6}$$

$$M_3 = i \begin{pmatrix} \frac{\partial}{\partial t} & 0 \\ 0 & -\frac{\partial}{\partial t} \end{pmatrix} \quad , \tag{1.3.7}$$

in the standard representation, we express the PDE as

$$L[\psi] = H\psi - mc\psi = 0 \quad , \tag{1.3.8}$$

where H is a 4-by-4 matrix such that

$$H = \begin{pmatrix} M_3 & M_2 \\ M_1 & M_3 \end{pmatrix} \quad . \tag{1.3.9}$$

Accordingly we have

$$Q(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \det H \tag{1.3.10}$$

$$= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right)^2 \quad , \tag{1.3.11}$$

and the characteristic determinant becomes

$$Q(\xi_\mu) = Q(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \tag{1.3.12}$$

$$= \left\{ \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 \right\}^2 \tag{1.3.13}$$

$$= (\xi_\mu \xi^\mu)^4 \quad . \tag{1.3.14}$$

Thus we find that the Dirac equations are hyperbolic, and the classical Dirac field is causal.

1.4 Causality for real Klein-Gordon fields

Now we treat causality in a quantum sense. Let us consider quantum real Klein-Gordon fields:

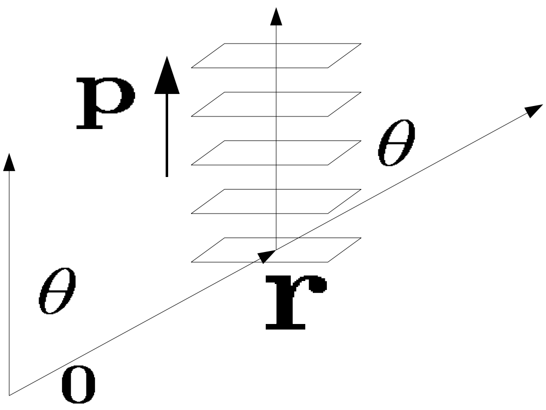
$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left\{ a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right\} \quad , \tag{1.4.1}$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ are destruction and creation operators of the scalar particles, respectively. We impose the conditions:

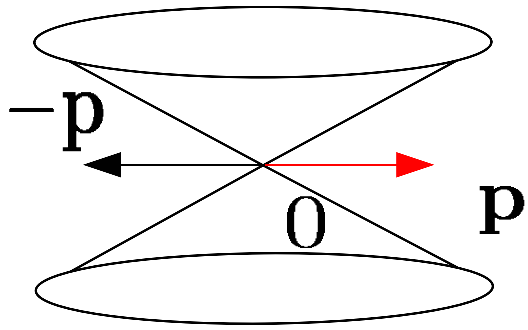
$$[a_{\mathbf{p}},\ a_{\mathbf{q}}^\dagger]_- = (2\pi)^3 \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [a_{\mathbf{p}},\ a_{\mathbf{q}}]_- = 0 \quad , \tag{1.4.2}$$

which suggests equal time commutation relations

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})]_- = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(t, \mathbf{x}), \phi(t, \mathbf{y})]_- = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})]_- = 0 \tag{1.4.3}$$



(a) The angle between 3-momentum $\mathbf{p}$ and the position vector $\mathbf{r}$



(b) An acausal motion $\mathbf{p}$ of a particle is cancelled out by the motion $-\mathbf{p}$ of its antiparticle.

Figure 4: The plane wave and its relation with the position vector in Fig 4a . The antiparticle cancels the particle's acausal motion in Fig 4b .

for these fields. The propagator for a real scalar field is

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} \quad , \tag{1.4.4}$$

and causality requires that, in a classical sense, the propagator should vanish for spacelike separations $(x-y)^2 = (x^\mu - y^\mu)(x_\mu - y_\mu) < 0$. However, it actually takes non-zero values. For example, in the case where $x^0 - y^0 = 0$, $\mathbf{x} - \mathbf{y} = \mathbf{r}$; this is a typically spacelike, indeed. The propagator becomes

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} & (1.4.5)\\ &= \int \frac{d|\mathbf{p}|}{(2\pi)^3}\frac{\mathbf{p}^2\sin\theta d\theta}{2E_{\mathbf{p}}} \times 2\pi \times e^{i|\mathbf{p}|r\cos\theta} & (1.4.6)\\ &= \frac{2\pi}{(2\pi)^3}\int_0^\infty d|\mathbf{p}|\frac{\mathbf{p}^2}{2E_{\mathbf{p}}}\frac{1}{i|\mathbf{p}|r}\left\{-e^{-i|\mathbf{p}|r\cos\theta} + e^{i|\mathbf{p}|r\cos\theta}\right\} & (1.4.7)\\ &= \frac{1}{(2\pi)^2 ir}\left[\int_0^\infty d|\mathbf{p}|\frac{|\mathbf{p}|}{2E_{\mathbf{p}}}e^{1|\mathbf{p}|r} - \int_0^{-\infty} d(-q)\frac{-q}{2E_q}e^{iqr}\right] \quad , & (1.4.8) \end{aligned}$$

where theta means the angle between the direction of the 3-momentum and the position vector (See Figure 4a), and in the second term of the last line, we put $|\mathbf{p}| = -q$. Changing the dummy variables, we have

$$D(x-y) = \frac{1}{(2\pi)^2 ir}\int_{-\infty}^{\infty} dz \frac{z}{2E_z} e^{izr} = \frac{1}{2(2\pi)^2 ir}\int_{-\infty}^{\infty} dz \frac{z}{\sqrt{z^2+m^2}} e^{izr} \quad , \tag{1.4.9}$$

and we need to evaluate this integral. Roughly,

$$D(x-y) \sim \frac{1}{r^2} \quad , \tag{1.4.10}$$

and it vanishes at large spatial distance, but it still takes non-zero value for spacelike separation.

How can we overcome this setback? In quantum mechanics, a measurement done at one point may affect a measurement at another point, which implies that information at the former point[14] travels to the latter point, and so our concern about causality protection is focused on whether or not the speed of travel of the information exceeds the speed of light. The statement that a measurement at the point $A = x^\mu$ does not make any effect on a measurement at the other point $B = y^\mu$ in which A and B is spacelike-separated (i.e. $(x^\mu - y^\mu)(x_\mu - y_\mu) < 0$) means that the propagation speed of the information is lower than the speed of light.

A postulate of quantum mechanics states that if two physical measurable quantities commute with each other, it is possible for us to measure them simultaneously. That is, commutativity between two observables assures that the one measurement does not affect the other measurement because possibility of simultaneous measurement may remove external factors, including the effect of the former measurement. In this case, the commutator is

$$[\phi(x), \phi(y)]_- = D(x-y) - D(y-x) \quad . \tag{1.4.11}$$

However we here have to investigate the commutativity at equal time because we now use[15] Heisenberg picture. Thus if an equal-time commutator $[\phi(\mathbf{x}), \phi(\mathbf{y})]_-$ vanishes for spacelike interval e.g. $(x^\mu - y^\mu)(x_\mu - y_\mu) < 0$, causality is protected. Commutators containing any function of $\phi(x)$ would also have to be zero.

In other words, for the purpose of studying the possibility of simultaneous measurement, we have only to consider the equal time commutation relation between them, because at equal time the interval

[14] It is also called signal velocity.

[15] It does not make sense if we consider a 'simultaneous' measurement between an operator $\mathcal{O}(t_0)$ and a time-evolved operator $\mathcal{O}(t_1)$.

is always spacelike: $(x-y) = (0, \mathbf{x}-\mathbf{y})$, $(x-y)^2 = -(\mathbf{x}-\mathbf{y})^2 < 0$. The equal time commutation relation for a real Klein-Gordon field is

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})]_- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \quad , \tag{1.4.12}$$

and here the left-hand-side is Lorentz invariant. The first term in the right-hand-side is rewritten as

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})} \tag{1.4.13}$$

$$\text{putting dummy variables as } \mathbf{p}' = -\mathbf{p} \quad \rightarrow = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \quad , \tag{1.4.14}$$

which leads to the cancellation of right-hand-side in (1.4.12) , and so it turns out that causality is protected.

1.5 Causality for complex Klein-Gordon fields

For the case of a massive complex scalar field,

$$\mathcal{L} = -(\partial_\mu\phi)(\partial^\mu\phi)^\dagger - m^2\phi\phi^\dagger \tag{1.5.1}$$

is a Lagrangian for this. Accordingly, the equations of motion

$$(\partial^2 - m^2)\phi = 0 \tag{1.5.2}$$

is derived from the Euler-Lagrange equation. Quantising this field, we have

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left\{a_{\mathbf{p}}e^{-ip\cdot x} + b^\dagger_{\mathbf{p}}e^{ip\cdot x}\right\} \tag{1.5.3}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left\{a^\dagger_{\mathbf{p}}e^{ip\cdot x} + b_{\mathbf{p}}e^{-ip\cdot x}\right\} \quad . \tag{1.5.4}$$

By definition, $\pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} = -(\partial^0\phi)^\dagger = -(\dot{\phi})^\dagger$, and so

$$\pi(x) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left\{a^\dagger_{\mathbf{p}}e^{ip\cdot x} - b_{\mathbf{p}}e^{-ip\cdot x}\right\} \tag{1.5.5}$$

$$\pi^\dagger(x) = i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left\{a_{\mathbf{p}}e^{-ip\cdot x} - b^\dagger_{\mathbf{p}}e^{ip\cdot x}\right\} \tag{1.5.6}$$

are the correspondent fields. The fact that two different creation operators $a_{\mathbf{p}}^\dagger$, $b_{\mathbf{p}}^\dagger$ are used here implies that the particle is different from the antiparticle. One draws attention to the fact that basically two fields ϕ and $\phi^\dagger$ are different[16]. Now let us consider the causality protection of these fields. The equal-time commutation relations are

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]_- = [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger]_- = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1.5.7}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}]_- = [b_{\mathbf{p}}, b_{\mathbf{q}}]_- = [a_{\mathbf{p}}, b_{\mathbf{q}}]_- = 0 \quad , \tag{1.5.8}$$

which suggests that the equal time commutation relations are

$$[\phi(t,\mathbf{x}),\ \pi(t,\mathbf{y})]_- = -i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(t,\mathbf{x}),\ \pi^\dagger(t,\mathbf{y})]_- = 0 \tag{1.5.9}$$

$$[\phi(t,\mathbf{x}),\ \phi(t,\mathbf{y})]_- = [\pi(t,\mathbf{x}),\ \pi(t,\mathbf{y})]_- = 0 \quad . \tag{1.5.10}$$

Now one finds that

$$[\phi(t,\mathbf{x}),\ \phi^\dagger(t,\mathbf{y})]_- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{1.5.11}$$

$$(1.4.14) \to = 0 \quad , \tag{1.5.12}$$

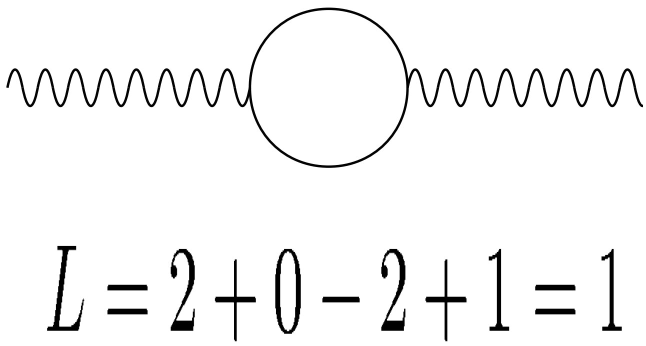
and so the information of these fields propagate in a causal way[17]. We can interpret this cancellation in (1.5.12) for the spacelike separation as a phenomenon in which once a particle behaves in an acausal manner, its antiparticle cancels it out (See Figure 4b). The same interpretation holds for the real Klein-Gordon field in (1.4.12) , which suggests that the scalar particle itself is its antiparticle. This interpretation is compatible with the existence of antiparticles, even though a scalar particle has not been observed so far.

2 Renormalizability of quantum theories

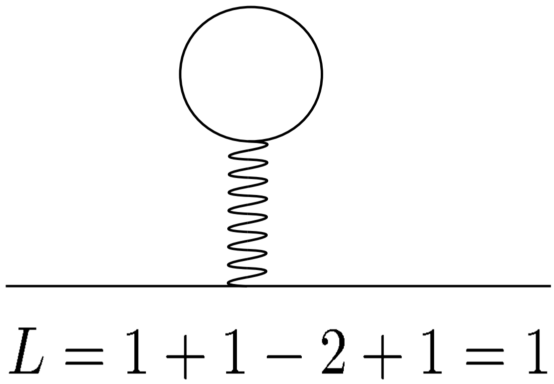
In QFT, whenever divergent terms appear, they should be removed or cancelled by technique. The series of technique to make divergent terms vanish are called renormalization. In this section, we conduct

[16]Therefore we can say that $\phi \neq \phi^\dagger$, that is, the field is not Hermitian, which means that two fields are not observable. Is it meaningless to discuss the causality of these fields?

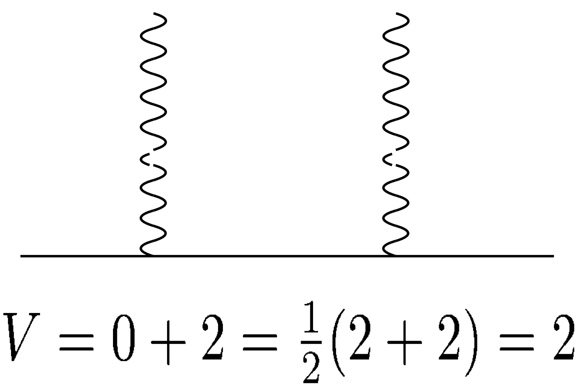
[17]Nevertheless is this result meaningful? Since the complex field ϕ is not an Hermitian operator, it is not measured basically.



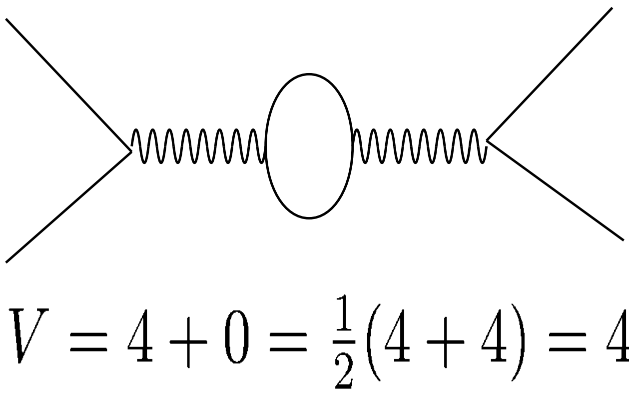
(a) two electron propagators, 2 external photon lines and 2 vertices



(b) one electron propagator, one photon propagator and two vertices



(c) two external photon lines, one electron propagator and two external electron propagators



(d) two photon propagators, two electron propagators and four electron external lines

Figure 5: examples of the formulae for the number of loops and the number of vertices

a brief review of renormalizability. To begin with, the power counting method is treated. It is seen that the (length) dimension of a coupling constant determines the renormalizability of a theory, and that QED is renormalizable in 4-dimensional space. Then the method of counter terms is reviewed, where renormalization conditions are introduced in renormalized perturbation theory. Feynman diagrams change into renormalized diagrams, and renormalization parameters are adjusted so as to satisfy the renormalization conditions, there. This section is finalised by study of one-loop diagrams for ϕ^4 theory.

In this section we use the following notation in terms of Feynman diagrams:

$$E_\gamma \text{ :the number of external lines of photon} \tag{2.0.13}$$
$$E_e \text{ :the number of external electron lines} \tag{2.0.14}$$
$$E_\phi \text{ :the number of external real scalar field lines} \tag{2.0.15}$$
$$P_\gamma \text{ :the number of propagators of photon} \tag{2.0.16}$$
$$P_e \text{ :the number of propagators of electron} \tag{2.0.17}$$
$$P_\phi \text{ :the number of propagators of real scalar field} \tag{2.0.18}$$
$$V \text{ :the number of vertices} \tag{2.0.19}$$
$$L \text{ :the number of loops} \tag{2.0.20}$$

2.1 Power-counting method

Firstly, the power-counting method for QED is considered. In a Feynman diagram, the number of loops is expressed as

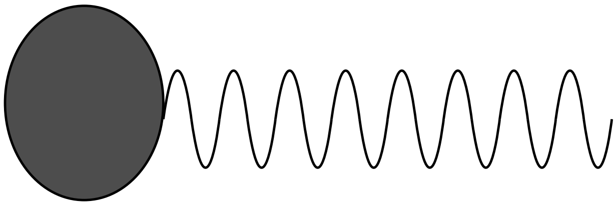
$$L = P_e + P_\gamma - V + 1 \quad , \tag{2.1.1}$$

since one electron propagator can give rise to one vertex and one loop, and one photon propagator is linked to two vertices. See Figure 5a and 5b, where two examples are cited. The number of vertices is

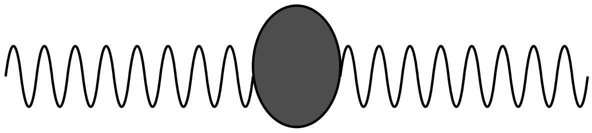
$$V = 2P_\gamma + E_\gamma = \frac{1}{2}\left(2P_e + E_e\right) \quad , \tag{2.1.2}$$

because one vertex line is linked[18] to one photon line and two electron lines. See Figure 5c and 5d , where two examples are cited. Now, for the purpose of investigate whether or not a Feynman diagram is

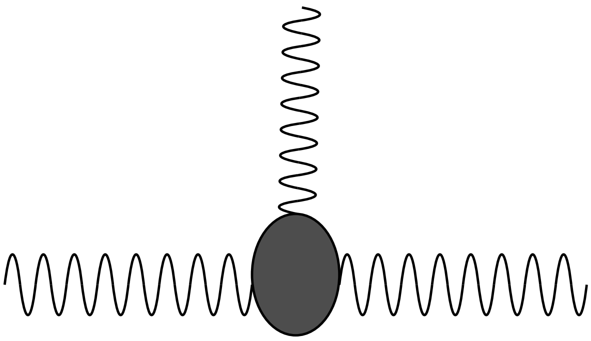
[18] One photon external line is linked to one vertex, and one photon propagator calls for two vertices. Similarly, in terms of electrons, one electron external line is linked to half of vertex, and each electron propagator is accompanied by substantially one vertex. These arguments are understood by drawing some Feynmann diagrams.



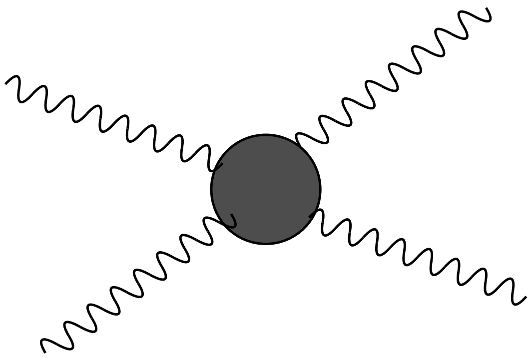
(a) D = 3, it vanishes due to Furry's theorem



(b) D = 2



(c) D = 1



(d) D = 0

Figure 6: Four diagrams among six fundamental and relevant diagrams in which $D \geq 0$ are shown. Each circle which is painted grey represents the total of all allowable diagrams.

divergent, let us introduce a concept of *superficial degree of divergence* D for its corresponding integral. Basically the integral is done with regard to momentum p in momentum space Feynman rule, and so

$$D = (\text{power of p in numerator}) - (\text{power of p in denominator}) \tag{2.1.3}$$

becomes the definition of the nomenclature. It is thought that when $D \geq 0$ for a Feynman diagram, the diagram diverges superficially, even though it is more complicated to study whether a diagram substantially diverges or not. Roughly, we can say that if $D > 0$, a diagram behaves as $e^V \Lambda^D$ in high energy region. Accordingly if $D = 0$, a diagram behaves as $e^V \ln \Lambda$, and when $D < 0$, a diagram converges superficially. Here Λ means a momentum cut-off. Now as a specific expression,

$$D = 4L - P_e - 2P_\gamma \tag{2.1.4}$$

is derived. Taking advantage of (2.1.1) and (2.1) we can rewrite the superficial degree of divergence as

$$\begin{aligned} D &= 4(P_e + P_\gamma - V + 1) - P_e - 2P_\gamma && (2.1.5)\\ &= 3P_e - 6P_\gamma - 4E_\gamma + 4 && (2.1.6)\\ &= 3E_\gamma + \frac{-3}{2}E_e + 4 - 4E_\gamma && (2.1.7)\\ &= 4 - E_\gamma + \frac{-3}{2}E_e \quad , && (2.1.8)\end{aligned}$$

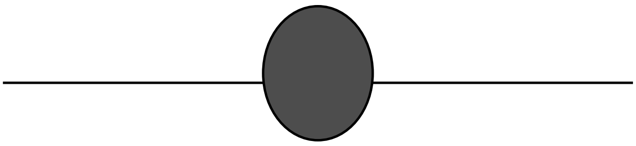
and so the superficial degree of divergence of a QED diagram is expressed only by external photon and electron lines. For a general d dimension, however, D contains V term. The formula for D is

$$D = dL - P_e - 2P_\gamma \quad , \tag{2.1.9}$$

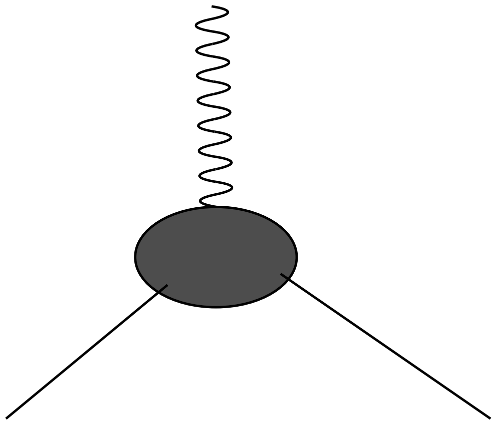
in d dimension, while the formulae (2.1.1) and are still valid in this case. After a simple calculation,

$$D = d + \frac{d-4}{2}V + \frac{1-d}{2}E_e + \frac{2-d}{2}E_\gamma \tag{2.1.10}$$

is obtained. It is seen that a 4-dimensional Minkowski space is special for the superficial degree of divergence. For the lower dimensional case than 4 dimension, since D decreases as the number of vertices increases, corresponding diagrams may converge. By contrast, in case of $d > 4$, the higher order a diagram becomes, the larger its D becomes, and so every diagram appears to be infinite-valued at a sufficiently high order in perturbation methods.



(a) D = 1



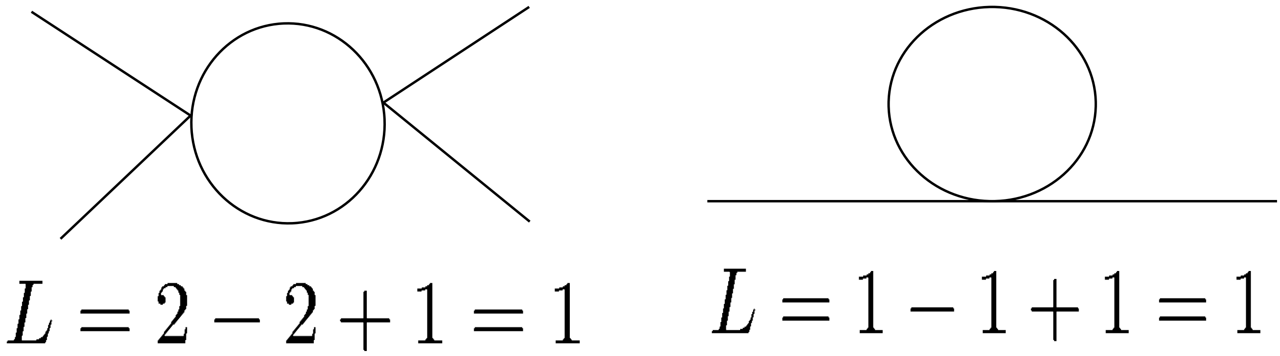
(b) D = 0

Figure 7: The remaining two among six fundamental and relevant diagrams in which $D \geq 0$ are shown. Each circle which is painted grey represents the total of all allowable diagrams.

Quantum theories are divided into three theories in terms of their behaviours in high energy region: super-renormalizable theory, renormalizable theory and non-renormalizable theory. A super-renormalizable theory is the theory in which only a finite number of Feynman diagrams diverge superficially. A renormalizable theory is the theory in which even though divergence happens at all orders in perturbation methods, the number of divergent diagrams is still finite. A non-renormalizable theory is the theory in which at sufficiently high order all diagrams diverge.

QED is a super-renormalizable theory in less than four dimension, a renormalizable theory in four dimension and a non-renormalizable theory in more than four dimension. It is because in QED only three fundamental diagrams diverge. To see it, here amputated and one-particle irreducible diagrams are considered, which suggests that our concern should be focused on fundamental and relevant six diagrams for QED. See Figure 6 and 7, where these diagrams are shown. These diagrams may diverge, and other diagrams which contain one of these diagrams may diverge. Furry's theorem states that any diagram of a fermion loop connecting an odd number of photons is zero, and accordingly we find that the diagrams of Figure 6a and 6c may vanish. As to the diagram of Figure 6d , the Ward identity requires its divergent part to be cancelled. The other three diagrams diverge logarithmically. Eventually, we find that only three primitive diagrams are divergent in QED.

From the view of dimensional analysis, we already know that the dimension of electron is $[e] = \frac{d-4}{2}$ (See



(a) 2 propagators, 2 vertices (b) 1 propagator, 1 vertex

Figure 8: Some examples for the formula of the number of loops in real scalar field theory

Appendix Dimensional analysis), and it is worthwhile to note that this value is equal to the coefficient of V in (2.1.10) . This fact suggests that renormalizability should depend on the dimension of the coupling constant. Superficial judgement for renormalizability may be formulated as

$$\text{If the dimension of coupling constant is negative, then the theory is super-renormalizable.} \tag{2.1.11}$$

$$\text{If the coupling constant is dimensionless, then the theory is renormaizable.} \tag{2.1.12}$$

$$\text{If the dimension of coupling constant is positive, then the theory is non-renormalizable.} \quad , \tag{2.1.13}$$

and so QED is renormalizable in 4-dimensional case.

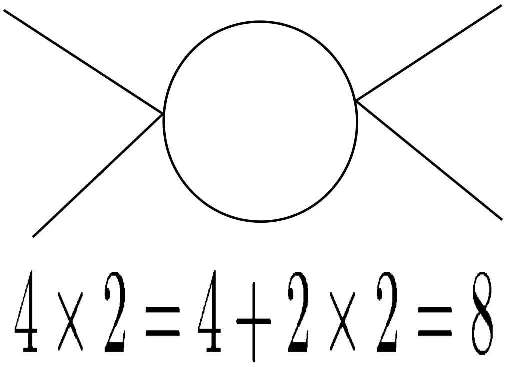
In this point of view, quantum linear gravity is a non-renormalizable theory because the coupling constant has the positive dimension: $[G] = 2$ (See Appendix Dimensional analysis).

Renormalizability for real scalar field theory: Secondly we consider the renormalizability of real scalar field theory in d dimension. The Lagrangian density is

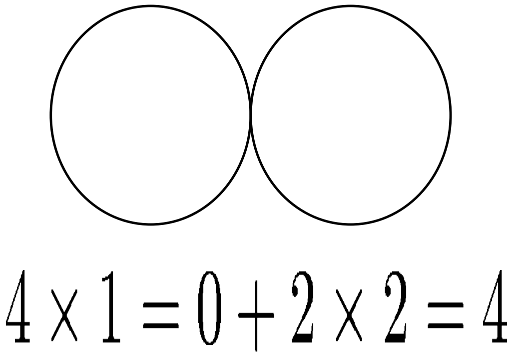
$$\mathcal{L}_{RSFT} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n \quad , \tag{2.1.14}$$

where its interaction is assumed to be an n-point, self-interaction. In a Feynamn diagram, the number of loop is

$$L = P_\phi - V + 1 \quad , \tag{2.1.15}$$



(a) 2 vertices, 2 propagators, 4 external lines



(b) 1 vertex, 2 propagators

Figure 9: Two examples for the formula $nV = E_\phi + 2P_\phi$ for ϕ^4 type interaction

since a loop is created at least with one propagator, and one propagator is accompanied by one net vertex. See Figure 8 , where two examples are shown. Each external leg has one combining point, while each propagator has two combining points, and so we have

$$nV = E_\phi + 2P_\phi \quad , \tag{2.1.16}$$

and two examples are shown in Figure 9 for the case of ϕ^4-type interaction. By using (2.1.15) and (2.1.16) ,

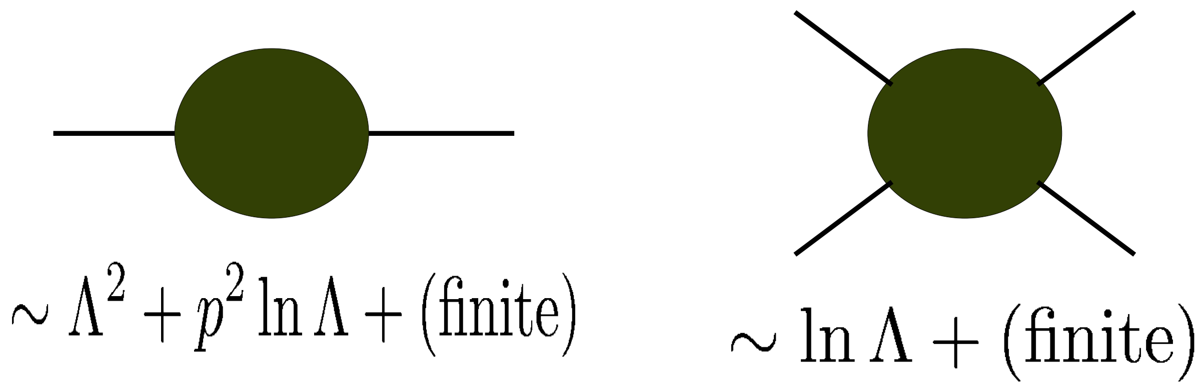
$$\begin{aligned} D &= dL - 2P_\phi & (2.1.17) \\ &= d + \frac{dn - 2d - 2n}{2} V + \frac{2-d}{2} E_\phi & (2.1.18) \end{aligned}$$

is deduced. Accordingly we find that for a 4-dimensional case ϕ^4 interaction is renormalizable.

2.2 Renormalization conditions

As a way of renormalization, counter terms are used. For example, for a ϕ^4 theory,

$$\begin{aligned} \mathcal{L} = &\left\{ \frac{1}{2}(\partial_\mu \phi_r)(\partial^\mu \phi_r) - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \right\} \\ &+ \left\{ \frac{1}{2}\delta_Z (\partial_\mu \phi_r)(\partial^\mu \phi_r) - \frac{1}{2}\delta_m \phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \right\} \end{aligned} \tag{2.2.1}$$



(a) One divergent diagram (b) One divergent diagram

Figure 10: The relevant divergent diagrams in ϕ^4 theory.

is a rescaled Lagrangian, and the second brace in the right-hand side of (2.2.1) contains the counter terms. Here the scalar field ϕ , its observable mass m and the observable ϕ^4 self-coupling constant λ are renormalized by the relations:

$$\phi = \sqrt{Z}\phi_r \tag{2.2.2}$$

$$\delta_m = m_0^2 Z - m^2 \ , \quad \delta_\lambda = \lambda_0 Z^2 - \lambda \quad , \tag{2.2.3}$$

where m_0 , λ_0 are renormalized quantities. ϕ_r is rescaled. The physically-relevant divergent diagrams for the ϕ^4 theory are shown in Figure 10 . They are cancelled out by these counter terms. As a result divergence in a renormalizable quantum field theory never appear[8] in measurable quantities.

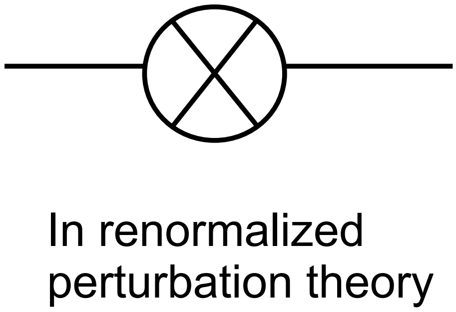
Now we have a problem. How do we define coupling constants? The answer is renormalization conditions, that is, in this case we have[19]

$$\text{the diagram in Figure 10a } = \frac{i}{p^2 - m^2} + (\text{terms regular at } \ p^2 = m^2) \tag{2.2.4}$$

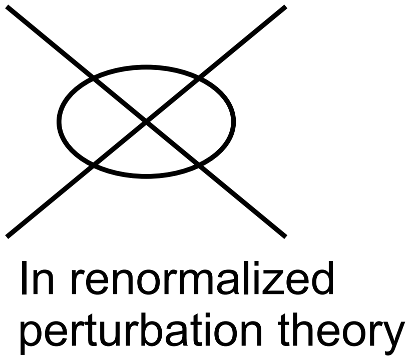
$$\text{the diagram in Figure 10b } = -i\lambda \text{at } \ s = 4m^2, t = u = 0 \quad . \tag{2.2.5}$$

By introducing these renormalization conditions, the Feynman rules are modified: Figure 10a and Figure 10b are replaced by Figure 11a and Figure 11b , respectively. In the modified Feynman rule, using

[19]In fact two conditions are contained in (2.2.4) . Let us also note that the diagram in Figure 10b is an amputated diagram.



(a) One modified diagram in renormalized perturbation theory



(b) One modified diagram in renormalized perturbation theory

Figure 11: The modified Feynman diagrams by renormalized perturbation theory in ϕ^4 case.

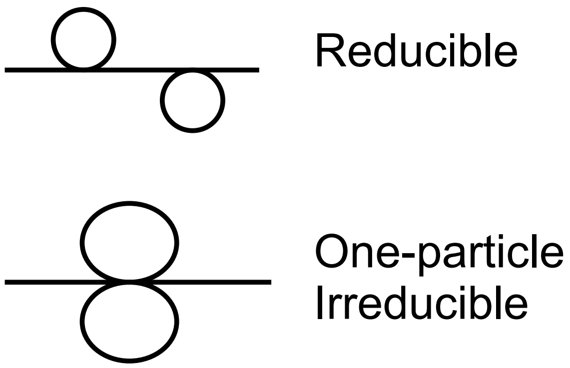
regulators, we have to adjust the parameters $\delta_\lambda,\ \delta_m,\ \delta_Z$ to satisfy the renormalization conditions. The amplitude does not depend on the regulator, and is finite after we adjust them. This method, in which modified Feynman rules stemming from counter terms are used as a c, is called renormalized perturbation theory. Before proceeding to the next section, let us refer to a one-particle irreducible diagram (1PI). Here a 1PI is defined as a diagram that cannot be separated into two diagrams by removing a single line. In Figure 12a , the upper diagram is reducible, while the lower is one-particle irreducible.

2.3 Calculation at one-loop level in ϕ^4 theory

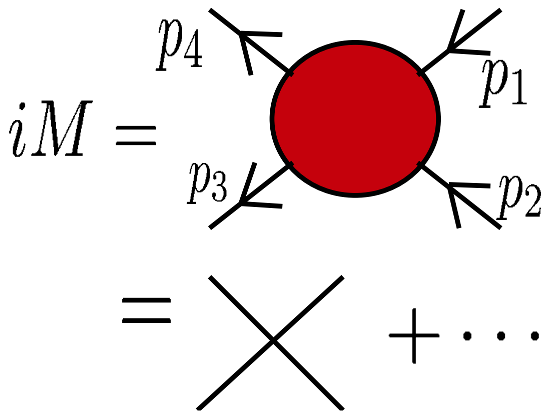
We now calculate the one-loop diagram for ϕ^4 theory. For example, let us consider a typical diagram for scattering of 2 particles, where there are 4 external lines (See Figure 12). There are three types of one-loop diagrams and a diagram of counter terms in Figure 12c , and first we consider one (Figure 12d) of them. Putting $p = p_1 + p_2$, we have

$$\text{diagram of Figure 12d } = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k+p)^2-m^2} \tag{2.3.1}$$

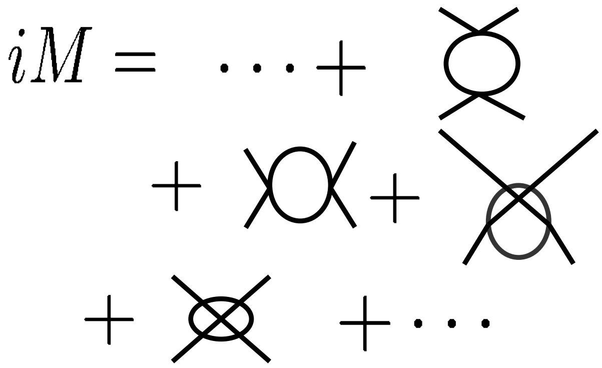
$$\equiv (-i\lambda)^2 \times iB(p^2) \quad . \tag{2.3.2}$$



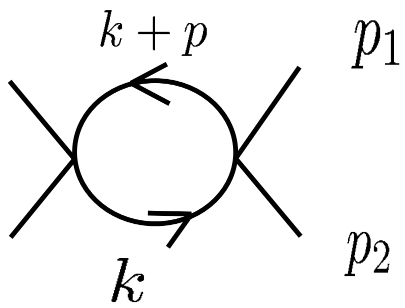
(a) Examples of a one-particle reducible and of an irreducible diagram



(b) The scattering amplitude for 2 particles in ϕ^4 theory



(c) Contributions to the scattering amplitude iM from the second and higher orders. The diagram for counter terms also appears.



(d) A diagram in the second order. This is one of one-loop diagrams.

Figure 12: An example of 1PI (in Fig 12a), and diagrams for 2-particle scattering in ϕ^4 theory (in Fig 12b , 12c and 12d)

Specifically, $B(p^2)$ is expressed as

$$B(p^2) = \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+2xk\cdot p+xp^2-m^2)^2} \tag{2.3.3}$$

$$= -\frac{1}{2}\int_0^1 dx \frac{\Gamma(\frac{4-d}{2})}{(4\pi)^{\frac{d}{2}}}\frac{1}{(m^2-x(1-x)p^2)^{\frac{4-d}{2}}} \tag{2.3.4}$$

$$\text{putting } d=4-\epsilon \to -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon}-\gamma+\ln 4\pi-\ln\left\{m^2-x(1-x)p^2\right\}\right) \tag{2.3.5}$$

$$\text{as } \ d\to 4 \quad , \tag{2.3.6}$$

where $\Gamma(x)$ is a gamma function and γ is the Euler-Mascheroni constant.

The other two one-loop diagrams are same if we change the Mandelstam variables s,t,u between them. The total amplitude at one-loop level is

$$iM = -i\lambda + (-i\lambda)^2\left\{iB(s)+iB(t)+iB(u)\right\} - i\delta_\lambda \quad , \tag{2.3.7}$$

and the renormalization conditions (2.2.4) and (2.2.5) demand that $iM=-i\lambda$ at $(s,t,u)=(4m^2,0,0)$. It follows[20] that

$$\delta_\lambda = -\lambda^2\left\{B(4m^2)+2B(0)\right\} \quad . \tag{2.3.8}$$

using the expression (2.3.5) , we have

$$\delta_\lambda \to \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left\{\frac{6}{\epsilon}-3\gamma+3\ln 4\pi-\ln\left(m^2-4m^2x(1-x)-2\ln m^2\right)\right\} \tag{2.3.9}$$

$$d\to 4 \quad , \tag{2.3.10}$$

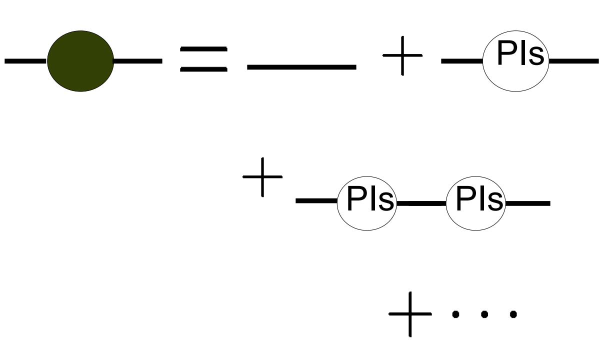
and this is divergent. However the total amplitude is finite:

$$iM = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1\left\{\ln\left(\frac{m^2-x(1-x)s}{m^2-4m^2x(1-x)}\right)+\ln\frac{m^2-x(1-x)t}{m^2}+\ln\left(\frac{m^2-x(1-x)u}{m^2}\right)\right\} \quad . \tag{2.3.11}$$

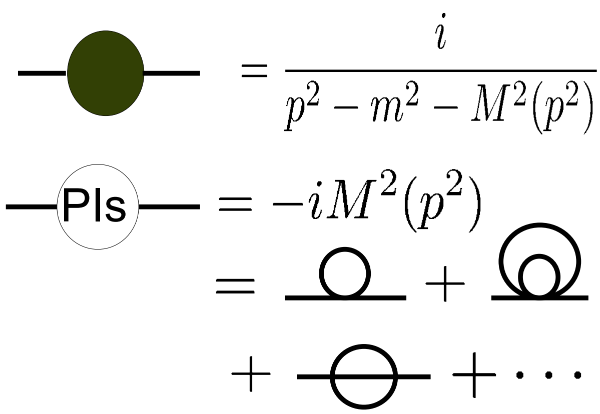
Next we use the renormalization conditions to determine $\delta_Z,\ \delta_m$. To see this we consider the two-leg function, and its perturbation expansion is expressed in Figure 13a , where the diagram with PIs is the sum of one-particle irreducible diagrams. Renormalization conditions are

$$M^2(p^2)\Big|_{p^2=m^2} = 0\ , \quad \frac{dM^2(p^2)}{dp^2}\Big|_{p^2=m^2} = 0 \quad . \tag{2.3.12}$$

[20] If we calculate higher-level loops, some correction terms are given to δ_λ .



(a) The perturbation expansion of the two-leg function



(b) The values of the two-leg function (upper) and of the one-particle-irreducible insertions (lower)

Figure 13: The two-leg function is expanded in a perturbation method in Fig 13b

Here we study the one-loop diagrams, and so the diagrams are in Figure 14a , and we have

$$-iM^2(p^2) = -\frac{i\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m) \tag{2.3.13}$$

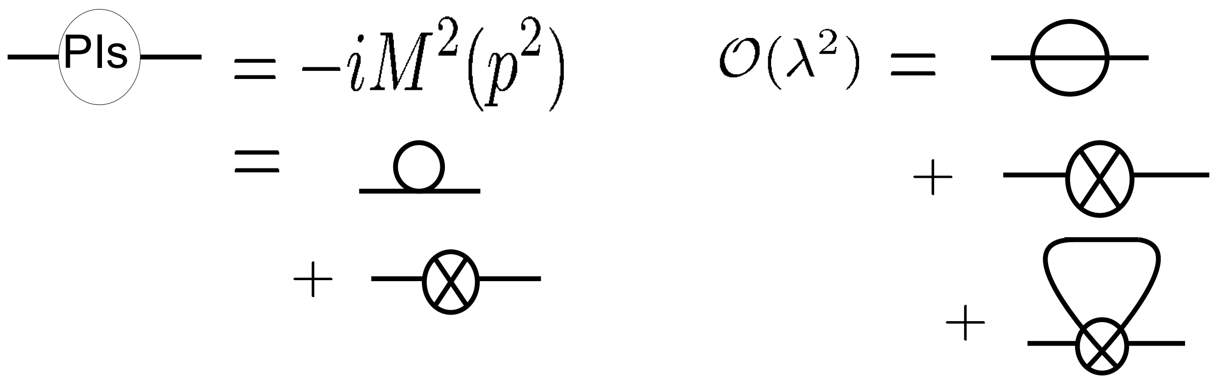
$$= -\frac{i\lambda}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}} + i(p^2\delta_Z - \delta_m) \quad . \tag{2.3.14}$$

We set

$$\delta_Z = 0\ , \quad \delta_m = -\frac{\lambda}{2(4\pi)^{\frac{d}{2}}}\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}} \quad , \tag{2.3.15}$$

which is compatible with the renormalizaion conditions (2.3.12) . The non-zero contributions to $M^2(p^2)$ start from the second order λ^2 , stemming from the diagrams in Figure 14b , where the third diagram has the δ_λ counter term. The second diagram is the $(p^2\delta_Z - \delta_m)$ counter term, and we adjust it so that the remaining divergences can cancel.

CHAPTER III



(a) The one-particle irreducible insertions at one-loop level

(b) The one-particle irreducible insertions at second order of the coupling constant

Figure 14: The one-particle irreducible diagrams at one-loop level (in Fig 14a) and at second order level of the coupling constant (in Fig 14b)

HIGHER-SPIN FIELDS

Fields with spin-0, spin-1/2 and spin-1 are described by Klein-Gordon, Dirac and Maxwell-Proca equations, respectively. QFT does not restrict itself within spin-1 fields. Spin-3/2 fields are ruled by Rarita-Schwinger equations, and equations of motion for spin-2 fields are linearised Einstein equations, even though corresponding elementary particles for these fields have not been discovered so far. In general, fields with more than spin-1 are called higher-spin fields.

This chapter discusses these higher-spin fields. Firstly, with the introduction of a rank-q generalised linear Christoffel symbol, equations of motions and Lagrangians for massless higher-spin fields are obtained. This derivation is based on the paper written by de Wit and Freedman[9] . Secondly, we see that in a classical level the Rarita-Schwinger field which couples with an external electromagnetic field violates causality. This discussion relies on the paper written by Velo and Zwanziger[10] , and Srokin's paper[11] is also referenced. This causality issue is also treated in the following section according to the paper written by Deser and Zumino[12] , where supergravity plays an important role in restoring cusality at a classical level.

In the fourth section, higher-spin fields are quantised with a generalised notion of a polarisation 4-

vector, and Feynman propagators are constructed in the momentum space. The methods of quantisation and of construction of Feynman propagator are based on the paper written by Huang et al[13] . In the fifth section, it turns out that higher-spin fields are non-renormalizable, which was reported by Deser and van Nieuwenhuizen[14] .

In the sixth section, Feynman rules are established for higher-spin fields.

3 Equations of motion for higher-spin fields

This section argues the equations of motion for massless higher-spin fields. In the first subsection, equations of motion for massless bosonic higher-spin fields are derived, where the notion of a linear Christoffel symbol is generalised. In the second subsection, the matter of gauge conditions and dynamical degrees of freedom are treated for bosonic fields. It is seen that the equations of motion are composed of not only evolution equations but also constraints equations. Lagrangians for these fields are also constructed. In the following section, similar discussion is conducted for fermionic fields. This section is closed with some explanation for symmetrised sum which is important for arguing equations of motion for these fields.

First of all, let us briefly refer to the equations of motion for massless higher-spin fields. Massless bosonic fields with spin-s are described by the following PDEs:

$$Z_{\mu_1\mu_2\ldots\mu_s}(x) = \partial^2\phi_{\mu_1\mu_2\ldots\mu_s} - \partial^\beta \sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \partial_{\mu_1}\phi_{\beta\mu_2\ldots\mu_s}(x) + \sum_{\mu,\mathrm{level}-2}^{\mathrm{sym}} \partial_{\mu_1}\partial_{\mu_2}\phi^{\beta}{}_{\beta\mu_3\ldots\mu_s}(x) = 0 \quad , \tag{3.0.16}$$

where the symbol $\sum^{\mathrm{sym}}$ denotes a symmetrised sum with respect to all non-contracted vector indices. For example, for completely symmetric rank-(s-1) tensor $B_{\mu_1\ldots\mu_{s-1}}$, we have

$$\sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \partial_{\mu_1}B_{\mu_2\mu_3\ldots\mu_s} \equiv \partial_{\mu_1}B_{\mu_2\ldots\mu_s} + \partial_{\mu_2}B_{\mu_1\mu_3\ldots\mu_s} + \cdots + \partial_{\mu_s}B_{\mu_1\mu_2\ldots\mu_{s-1}} \quad , \tag{3.0.17}$$

and we will discuss the properties of this symmetrised sum later. Obviously the massless case of the equations of motion (1.5.1) is compatible with (3.0.16) , and for the case of massless spin-1, we have

$$Z_\mu = \partial^2 A_\mu - \partial_\mu\partial_\nu A^\nu = 0 \quad , \tag{3.0.18}$$

this is precisely the Maxwell equations without electric current. Accordingly for the case of massless spin-2, the differential equations

$$Z_{\lambda\mu} = \partial^2 A_{\lambda\mu} - \{\partial_\lambda \partial^\nu A_{\nu\mu} + \partial_\mu \partial^\nu A_{\nu\lambda}\} + \partial_\lambda \partial_\mu A^\xi{}_\xi = 0 \tag{3.0.19}$$

is derived; this is a linearised Einstein equations (A.2.16) .

On the other hand, the equations of motion for massless fermionic fields $\{\psi^a_{\mu_1 \dots \mu_s}\}$ with spin-$(s+\frac{1}{2})$ are

$$Z^a_{\mu_1\mu_2\dots\mu_s} = (\gamma^\nu \partial_\nu \psi)^a_{\mu_1\mu_2\dots\mu_s} - \sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \partial_{\mu_1} \gamma^\nu \psi^a_{\nu\mu_2\mu_3\dots\mu_s} = 0 \quad , \tag{3.0.20}$$

and for a massless spin-1/2 field we have a massless Dirac equations from (3.0.20) .

3.1 Equations of motion for massless bosonic fields

In order to derive the equations of motion for these fields, firstly let us introduce a notion of a generalised linear Christoffel symbols: a rank-q linear Christoffel symbol $\Gamma^{(q)}_{\beta_1\dots\beta_q;\mu_1\dots\mu_s}$ for spin-s is defined as

$$\Gamma^{(q)}_{\beta_1\dots\beta_q;\mu_1\dots\mu_s} \equiv \partial_{\beta_1} \Gamma^{(q-1)}_{\beta_2\dots\beta_q;\mu_1\dots\mu_s} - \frac{1}{q} \sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \partial_{\mu_1} \Gamma^{(q-1)}_{\beta_2\dots\beta_q;\beta_1\mu_2\dots\mu_s} \tag{3.1.1}$$

$$\text{and } \Gamma^{(1)}_{\beta_1;\mu_1\dots\mu_s} = \partial_{\beta_1} \phi_{\mu_1\dots\mu_s} - \sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \partial_{\mu_1} \phi_{\beta_1\mu_2\dots\mu_s} \ , \quad \Gamma^{(0)}_{\mu_1\dots\mu_s} = \phi_{\mu_1\mu_2\dots\mu_s} \tag{3.1.2}$$

where s independent permutations of the $\{\mu_j\}$ are included in the summation. Obviously, the rank-1 linear Christoffel symbol for spin-s is symmetric with regard to the spin-related indices $\{\mu_i\}$. Then, provided that the rank-(q-1) linear Christoffel symbol is symmetric with regard to the spin-related indices, we can deduce, by the recursion relation above, that the rank-q linear Christoffel symbol is also symmetric. We, therefore, find that the rank-q linear Christoffel symbol is symmetric with regard to the spin-related indices $\{\mu_i\}$. It is assumed that in terms of a gauge transformation, the variation of the symbol is

$$\delta\Gamma^{(q)}_{\beta_1\dots\beta_q;\mu_1\dots\mu_s} = (-1)^q (q+1) \sum_{\mu,\mathrm{level}-(\mathrm{q}+1)}^{\mathrm{sym}} \partial_{\mu_1} \cdots \partial_{\mu_{q+1}} \zeta_{\beta_1\dots\beta_q\mu_{q+2}\dots\mu_s} \quad , \tag{3.1.3}$$

$$\text{and } \delta\Gamma^{(s)}_{\beta_1\dots\beta_s;\mu_1\dots\mu_s} = 0 \quad , \tag{3.1.4}$$

where a completely symmetric rank-(s-1) tensor $\zeta_{\mu_2 \ldots \mu_s}$ is a gauge parameter. For the purpose of keeping gauge invariance, we demand that the gauge parameter should be traceless:

$$\zeta^{\nu}{}_{\nu\mu_3\ldots\mu_s} = 0 \quad . \tag{3.1.5}$$

For example, an EM field A_μ its gauge variation is

$$\delta A_\mu = \delta\Gamma^{(0)}_\mu = \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\mu \zeta = \partial_\mu \zeta \quad , \tag{3.1.6}$$

and so this definition[21] (3.1.4) may be appropriate. Generally, from (3.1.2) and (3.1.3)

$$\delta\Gamma^{(0)}_{\mu_1\ldots\mu_s} = \delta\phi_{\mu_1\ldots\mu_s} = \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\zeta_{\mu_2\ldots\mu_s} \tag{3.1.7}$$

is derived. In addition, (3.1.4) assures that the rank-s linear Christoffel symbol for spin-s has gauge-invariance. The tensor

$$R_{\beta_1\ldots\beta_s;\mu_1\ldots\mu_s} \equiv \Gamma^{(s)}_{\beta_1\ldots\beta_s;\mu_1\ldots\mu_s} \tag{3.1.8}$$

is called a generalised linear Riemann curvature tensor for spin-s. For a massless scalar field, the generalised linear Riemann tensor is the field itself:

$$R = \Gamma^{(0)} = \phi \quad . \tag{3.1.9}$$

In the case of a massless vector field, the generalised linear Riemann tensor is written as

$$R_{\beta;\mu} = \Gamma^{(1)}_{\beta;\mu} = \partial_\beta\phi_\mu - \partial_\mu\phi_\beta \quad , \tag{3.1.10}$$

that is, the field strength tensor. For spin-2 field,

$$\begin{aligned} R_{\beta_1\beta_2;\mu_1\mu_2} &= \Gamma^{(2)}_{\beta_1\beta_2;\mu_1\mu_2} & (3.1.11)\\ &= \partial_{\beta_1}\partial_{\beta_2}h_{\mu_1\mu_2} + \partial_{\mu_1}\partial_{\mu_2}h_{\beta_1\beta_2} \\ &\quad + \frac{-1}{2}\left\{\partial_{\beta_1}\partial_{\mu_1}h_{\beta_2\mu_2} + \partial_{\beta_1}\partial_{\mu_2}h_{\beta_2\mu_1} + \partial_{\mu_1}\partial_{\beta_2}h_{\beta_1\mu_2} + \partial_{\beta_2}\partial_{\mu_2}h_{\beta_1\mu_1}\right\} & (3.1.12) \end{aligned}$$

is obtained[22] .

[21] The U(1) gauge transformation is $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\lambda$, and here we should see the parameters have the relation $\frac{1}{e}\lambda = \zeta$.

[22] This is same as the generalised linear Riemann curvature tensor (A.2.11) defined in the Appendix Dimensional analysis for field theories.

Rank-q linear Christoffel symbols for spin-s: According to (3.1.1) and (3.1.2) , we can compute the rank-2 and rank-3 linear Christoffel symbols directly

$$\Gamma^{(2)}_{\beta_1\beta_2;\mu_1\ldots\mu_s} = \partial_{\beta_1}\partial_{\beta_2}\phi_{\mu_1\ldots\mu_s} + \frac{-1}{2}\sum_{\beta,\text{level}-1}^{\text{sym}}\sum_{\mu,\text{level}-1}^{\text{sym}}\partial_{\beta_1}\partial_{\mu_1}\phi_{\beta_2\mu_2\ldots\mu_s} + \sum_{\mu,\text{level}-2}^{\text{sym}}\partial_{\mu_1}\partial_{\mu_2}\phi_{\beta_1\beta_2\mu_3\ldots\mu_s} \quad , \tag{3.1.13}$$

and also the rank-3 linear Christoffel symbol

$$\begin{aligned}\Gamma^{(3)}_{\beta_1\beta_2\beta_3;\mu_1\ldots\mu_s} = {} & \partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\phi_{\mu_1\ldots\mu_s} + \frac{-1}{3}\sum_{\beta,\text{level}-1}^{\text{sym}}\sum_{\mu,\text{level}-1}^{\text{sym}}\partial_{\mu_1}\partial_{\beta_2}\partial_{\beta_3}\phi_{\beta_1\mu_2\ldots\mu_s} \\ & + \frac{1}{3}\sum_{\beta,\text{level}-2}^{\text{sym}}\sum_{\mu,\text{level}-2}^{\text{sym}}\partial_{\beta_2}\partial_{\mu_1}\partial_{\mu_2}\phi_{\beta_1\beta_3\mu_3\ldots\mu_s} - \sum_{\mu,\text{level}-3}^{\text{sym}}\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi_{\beta_1\beta_2\beta_3\mu_4\ldots\mu_s} \quad .\end{aligned} \tag{3.1.14}$$

The rank-q linear Christoffel symbol for spin-s is written as

$$\begin{aligned}\Gamma^{(q)}_{\beta_1\ldots\beta_q;\mu_1\ldots\mu_s} = {} & \partial_{\beta_1}\cdots\partial_{\beta_q}\phi_{\mu_1\ldots\mu_s} + \sum_{j=1}^{q-1}\frac{(-1)^j}{{}_qC_j}\sum_{\beta,\text{level}-\text{j}}^{\text{sym}}\sum_{\mu,\text{level}-\text{j}}^{\text{sym}}\partial_{\mu_1}\cdots\partial_{\mu_j}\partial_{\beta_{j+1}}\cdots\partial_{\beta_j}\phi_{\beta_1\ldots\beta_j\mu_{j+1}\ldots\mu_s} \\ & + (-1)^q\sum_{\mu,\text{level}-\text{q}}^{\text{sym}}\partial_{\mu_1}\cdots\partial_{\mu_q}\phi_{\beta_1\ldots\beta_q\mu_{q+1}\ldots\mu_s} \quad .\end{aligned} \tag{3.1.15}$$

Equations of motion of a field with integer spin: Now we consider how equations of motion are derived for bosonic cases. By analogy with classical[23] theories, we make an assumption that the differential equations of motion for fields with integer spin-s are second-order. Taking advantage of the generalised linear Christoffel symbol, we have

$$Z_{\mu_1\ldots\mu_s} = \Gamma^{(2)\beta}{}_{\beta;\mu_1\ldots\mu_s} = \eta^{\beta_1\beta_2}\Gamma^{(2)}_{\beta_1\beta_2;\mu_1\ldots\mu_s} \tag{3.1.16}$$

$$= \eta^{\beta_1\beta_2}\left\{\partial_{\beta_1}\partial_{\beta_2}\phi_{\mu_1\ldots\mu_s} + \frac{-1}{2}\sum_{\beta,\text{level}-1}^{\text{sym}}\sum_{\mu,\text{level}-1}^{\text{sym}}\partial_{\beta_1}\partial_{\mu_1}\phi_{\beta_2\mu_1\ldots\mu_s}\sum_{\mu,\text{level}-2}^{\text{sym}}\partial_{\mu_1}\partial_{\mu_2}\phi_{\beta_1\beta_2\mu_3\ldots\mu_s}\right\} \tag{3.1.17}$$

$$= \partial^\beta\partial_\beta\phi_{\mu_1\ldots\mu_s} - \partial^\beta\sum_{\mu,\text{level}-1}^{\text{sym}}\partial_{\mu_1}\phi_{\beta\mu_2\ldots\mu_s} + \sum_{\mu,\text{level}-2}^{\text{sym}}\partial_{\mu_1}\partial_{\mu_2}\phi^\beta{}_{\beta\mu_3\ldots\mu_s} = 0 \quad , \tag{3.1.18}$$

that is, (3.0.16) is derived, and so these are equations of motion for a spin-s bosonic field. Then let us confirm that these equations of motion are gauge invariant. By (3.1.3)

$$\delta\Gamma^{(2)}_{\beta_1\beta_2;\mu_1\ldots\mu_s} = 3\sum_{\mu,\text{level}-3}^{\text{sym}}\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\zeta_{\beta_1\beta_2\mu_4\ldots\mu_s} \tag{3.1.19}$$

[23]Maxwell equations are second-order partial differential equations, and equations of motion for particles are described by Hamilton-Jacobi equations.

and

$$\eta^{\beta_1\beta_2}\delta\Gamma^{(2)}_{\beta_1\beta_2;\mu_1\ldots\mu_s} = 3\sum_{\mu,\text{level}-3}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\zeta^{\beta}{}_{\beta\mu_4\ldots\mu_s} = 0 \tag{3.1.20}$$

is derived because of (3.1.5) . Accordingly we have

$$\delta Z_{\mu_1\ldots\mu_s} = \delta\Gamma^{(2)\beta}{}_{\beta;\mu_1\ldots\mu_s} = \eta^{\beta_1\beta_2}\delta\Gamma^{(2)}_{\beta_1\beta_2;\mu_1\ldots\mu_s} = 0 \quad , \tag{3.1.21}$$

and therefore the equations of motion for higher-spin fields are gauge invariant.

3.2 Gauge conditions for massless bosonic fields

Maxwell equations allow us to take a gauge condition to fix a gauge. For an arbitrary spin-s bosonic field $\phi_{\mu_1\ldots\mu_s}$, the gauge-fixing for their equations is conducted as

$$G_{\mu_2\ldots\mu_s} \equiv \partial^{\lambda}\phi_{\lambda\mu_2\ldots\mu_s} + \frac{-1}{2}\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_2}\phi^{\lambda}{}_{\lambda\mu_3\ldots\mu_s} = 0 \quad , \tag{3.2.1}$$

and in these conditions the equations of motion become

$$Z_{\mu_1\ldots\mu_s} = \partial^{\epsilon}\partial_{\epsilon}\phi_{\mu_1\ldots\mu_s} = \partial^2\phi_{\mu_1\ldots\mu_s} = 0 \quad , \tag{3.2.2}$$

which suggests that its gauge invariance should be conserved in this gauge condition. We saw the gauge invariance of these fields in last subsection. Now let us reconfirm it. If $\partial^2\phi_{\mu_1\ldots\mu_s} = 0$, then we take the gauge variation for both sides;

$$\delta\partial^2\phi_{\mu_1\ldots\mu_s} = \partial^2\delta\phi_{\mu_1\ldots\mu_s} = \partial^2\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\zeta_{\mu_2\ldots\mu_s} = 0 \quad , \tag{3.2.3}$$

where we use (3.1.7) . We take the gauge variation for both sides (3.2.1) :

$$\delta G_{\mu_2\ldots\mu_s} = \delta\partial^{\lambda}\phi_{\lambda\mu_2\ldots\mu_s} + \delta\frac{-1}{2}\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_2}\phi^{\lambda}{}_{\lambda\mu_3\ldots\mu_s} = 0 \tag{3.2.4}$$

$$\Leftrightarrow \delta G_{\mu_2\ldots\mu_s} = \partial^{\beta}\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\beta}\zeta_{\mu_2\ldots\mu_s} + 0 = 0 \tag{3.2.5}$$

$$\Leftrightarrow \delta G_{\mu_2\ldots\mu_s} = \partial^{\beta}\partial_{\beta}\zeta_{\mu_2\ldots\mu_s} + \partial^{\beta}\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_2}\zeta_{\beta\mu_3\ldots\mu_s} = 0 \tag{3.2.6}$$

$$\Leftrightarrow \delta G_{\mu_2\ldots\mu_s} = \partial^{\beta}\partial_{\beta}\zeta_{\mu_2\ldots\mu_s} + 0 = 0 \quad , \tag{3.2.7}$$

and therefore we have $\partial^\nu\partial_\nu\zeta_{\mu_2\ldots\mu_s} = \partial^2\zeta_{\mu_2\ldots\mu_s} = 0$ and accordingly

$$\delta Z_{\mu_1\ldots\mu_s} = \delta\partial^2\phi_{\mu_1\ldots\mu_s} = \partial^2 \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\zeta_{\mu_2\ldots\mu_s} = 0 \quad , \tag{3.2.8}$$

we see that gauge invariance is valid under this gauge-fixing condition. The result (3.2.7) , that is, $\partial^2\zeta_{\mu_2\ldots\mu_s} = 0$ implies that certain components of $\phi_{\mu_1\ldots\mu_s}$ still can be regauged without breaking the gauge condition (3.2.1) as far as $\partial^2\zeta_{\mu_2\ldots\mu_s} = 0$ is valid. For spin-1 and spin-2 fields, the gauge-fixing conditions (3.2.1) reduce to

$$G = \partial^\mu A_\mu = 0\ , \quad \partial^2\zeta = 0 \tag{3.2.9}$$

$$G_\mu = \partial^\lambda h_{\lambda\mu} - \frac{1}{2}\partial_\mu h^\lambda_{\ \lambda} = 0\ , \quad \partial^2\zeta_\mu = 0 \tag{3.2.10}$$

respectively. These are familiar Lorentz and de Donger gauge conditions.

Dynamical degrees of freedom for massless bosonic fields: For spin-1 and spin-2 massless fields, they have 4 and 10 independent components, respectively. In general, a rank-s completely symmetric tensor field $\phi_{\mu_1\ldots\mu_s}$ in d dimension has ${}_{s+d-1}C_{d-1}$ independent components[24]. Now we impose constraints for the spin-s fields $\phi_{\mu_1\ldots\mu_s}$, $(s \geq 4)$:

$$\phi^{\kappa\lambda}{}_{\kappa\lambda\mu_5\ldots\mu_s} = 0 \tag{3.2.11}$$

these are, what is called, the double-traceless conditions. Thus the number of independent components are[25]

$${}_{s+4-1}C_{4-1} - {}_{s+4-5}C_{4-1} = 2s^2 + 2 \tag{3.2.12}$$

for 4-dimensional cases. Furthermore, we use the gauge-fixing conditions, which have s^2 independent conditions. In addition, imposing these gauge conditions, we choose the field's gauge parameters so as to regauge s^2 components. Then the only remaining independent components of the massless bosonic field

[24] For a d-dimensional case, we use d independent indices. Provided that we use digit 1 n_1 times, digit 2 n_2 times, ... and digit d n_d times, then we compute the total number of possible combinations $(n_1, \ldots, n_{d-1}, n_d)$ which satisfies $n_1 + \cdots n_d = s$, $0 \leq n_k \leq s$. Accordingly we obtain the quantity ${}_{s+d-1}C_{d-1}$.

[25] Let us emphasise that (3.2.12) is for 4-dimensional case.

is two. This result is suitable for describing massless fields. Indeed, transverse[26] traceless components of such a massless field are s-fold tensor products of transverse polarisation vectors:

$$\epsilon_{\lambda}^{\mu_1}(k)\epsilon_{\lambda}^{\mu_1}(k)\cdots\epsilon_{\lambda}^{\mu_s}(k)e^{-ik\cdot x} \tag{3.2.13}$$

$$\text{and, } \epsilon_{-\lambda}^{\mu_1}(k)\epsilon_{-\lambda}^{\mu_1}(k)\cdots\epsilon_{-\lambda}^{\mu_s}(k)e^{-ik\cdot x} \quad , \tag{3.2.14}$$

where helicity $\pm s$ are carried. Later we will see that the polarisation vector is generalised for the purpose of quantising fields.

Furthermore the equations of motion $Z_{\mu_1\ldots\mu_s}=0$ may consist of evolution equations (with second-order time derivatives) and constraints equations (with at most first-order time derivatives) on the initial data. Let us adopt

$$Z_{0j_2\ldots j_s}=0 \tag{3.2.15}$$

$$Z_{00j_3\ldots j_s}-Z_{iij_3\ldots j_s}=0 \tag{3.2.16}$$

$$i,j_2,\ldots j_s=1,2,3 \tag{3.2.17}$$

as the constraints. (3.2.15) make ${}_{s+1}C_2=\frac{s(s+1)}{2}$ constraints, while (3.2.16) do ${}_sC_2=\frac{s(s-1)}{2}$ constraints, and so we have totally s^2 constraints. Since the number of the independent components of the gauge parameter $\zeta_{\mu_2\ldots\mu_s}$ is ${}_{s+2}C_3-{}_sC_3=s^2$, this number is the same as that of constraints. However, once we take the gauge conditions (3.2.1) , the constraints equations become evolution equations, that is the d'Alembertian equations (3.2.2) . Hence we need not care the constraints in the gauge conditions.

Lagrangian for massless bosonic fields: For a higher-spin field, some constraints are necessary for keeping gauge invariance and for constructing its Lagrangian density. For example, the Lagrangian density[27] for QED is expressed with $F_{\mu\nu}$, that is, $R_{\mu;\nu}=\Gamma^{(1)}_{\mu;\nu}$ in the new notation (3.1.10) . It is expected

[26]If $\phi_{i_1\ldots i_s}$ is the transverse component of the field in momentum space, $k^j\phi_{ji_2\ldots i_s}=0$, where $j,i_1\ldots i_s$ are 2-valued indices because it is on 2-dimensional space. By (3.2.1) , we have $\phi_{jji_3\ldots i_s}=0$.

[27]See Appendix Dimensional analysis for field theories.

that $\Gamma^{(1)}_{\beta;\mu_1\ldots\mu_s}$ form the Lagrangian density for a spin-s ($s \geq 1$) bosonic field.

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{16\pi}\Gamma^{(1)}_{\beta;\mu}\Gamma^{(1)\beta;\mu}\delta_{s,1} \\
&+ (1-\delta_{s,1}) \times \frac{1}{64\pi}\left[\frac{1}{2(s-1)}\Gamma^{(1)\beta;\mu_1\ldots\mu_s}\left\{s\Gamma^{(1)}_{\mu_1;\beta\mu_2\ldots\mu_s} - (s-2)\Gamma^{(1)}_{\beta;\mu_1\ldots\mu_s}\right\}\right. \\
&+ \frac{s}{2(s-1)}\Gamma^{(1);\beta\mu_2\ldots\mu_s}_{\beta}\left\{(s-2)\Gamma^{(1)\beta}{}_{;\beta\mu_2\ldots\mu_s} - (s-1)\Gamma^{(1)\beta}_{\mu_2;}{}_{\beta\mu_3\ldots\mu_s}\right\} \\
&+ \frac{1}{8}s(s-2)\Gamma^{(1)}_{\beta_1;\beta_2}{}^{\mu_3\ldots\mu_s}\left\{\Gamma^{(1)\beta_1;\beta_2}{}_{\beta_2\mu_3\ldots\mu_s} - \Gamma^{(1)}_{\mu_3;\beta_2}{}^{\beta_2\beta_1}{}_{\mu_4\ldots\mu_s}\right\} \\
&\left.+\frac{1}{16}s(s-1)(s-2)\Gamma^{(1)}_{\beta_1;\beta_2}{}^{\beta_1\beta_2\mu_4\ldots\mu_s}\Gamma^{(1)\beta_1;\beta_2}{}_{\beta_1\beta_2\mu_4\ldots\mu_s}\right]
\end{aligned} \tag{3.2.18}
$$

Indeed, for a massless spin-1 field, we have

$$
\mathcal{L} = -\frac{1}{16\pi}\Gamma^{(1)}_{\beta;\mu}\Gamma^{(1)\beta;\mu} = \frac{-1}{16\pi}\left(\partial_\beta A_\mu - \partial_\mu A_\beta\right)\left(\partial^\beta A^\mu - \partial^\mu A^\beta\right) \quad . \tag{3.2.19}
$$

For a spin-2 field, we have

$$
\mathcal{L} = \frac{1}{64\pi}\left\{\Gamma^{(1)\beta;\mu_1\mu_2}\Gamma^{(1)}_{\mu_1;\beta\mu_2} - \Gamma^{(1)}_{\beta;}{}^{\beta\mu_2}\Gamma^{(1)}_{\mu_2;\beta}{}^{\beta}\right\} \quad , \tag{3.2.20}
$$

but in the de Donger gauge and h^λ_λ conditions, $\Gamma^{(1)}_{\beta;}{}^{\beta\mu_2}$ vanish. Thus the Lagrangian density for a massless spin-2 field is

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{64\pi}\Gamma^{(1)\beta;\mu_1\mu_2}\Gamma^{(1)}_{\mu_1;\beta\mu_2} && (3.2.21) \\
(\text{A.2.7}) \rightarrow &= \frac{1}{64\pi} \times 4\Gamma^{\beta\mu_1\mu_2}\Gamma_{\mu_1\beta\mu_2} && (3.2.22) \\
&= \frac{1}{16\pi}\Gamma^{\beta\mu_1\mu_2}\Gamma_{\mu_1\mu_2\beta} && (3.2.23) \\
(\text{A.2.6}) \rightarrow &= \frac{1}{16\pi}\Gamma^{\beta\mu_1\mu_2}\Gamma_{\mu_2\mu_1\beta} && (3.2.24)
\end{aligned}
$$

and accordingly the action of the field, with restored physical constant G , is

$$
S_g = \frac{1}{16\pi G}\int d^4x\Gamma^{\beta\mu_1\mu_2}\Gamma_{\mu_2\mu_1\beta} \quad , \tag{3.2.25}
$$

and this is consistent with (A.2.18) .

3.3 Equations of motion for massless fermionic fields

Now we derive the equations of motion for a field with a half-integer spin. Concerning a massless spin-($s+\frac{1}{2}$) field $\psi^a_{\mu_1\ldots\mu_s}$ which is a totally symmetric rank-s tensor-spinor[28] , the (infinitesimal) gauge

[28]The index a is a spinor index.

transformation is

$$\delta\psi^a_{\mu_1\ldots\mu_s} = \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\kappa^a_{\mu_2\mu_3\ldots\mu_s} \quad , \tag{3.3.1}$$

where the gauge parameter $\kappa^a_{\mu_2\mu_3\ldots\mu_s}$ is a totally symmetric rank-(s-1) tensor-spinor, and it satisfies the condition:

$$\gamma^\nu\kappa^a_{\nu\mu_3\ldots\mu_s} = 0 \quad , \tag{3.3.2}$$

that is, the gauge parameters are traceless regarding the gamma matrices. Suppose that the equations of motion are first-order PDEs. Since the equations of motion should be gauge invariant, rank-1 generalised linear Christoffel symbols are used with gamma matrices. The only possible combination is

$$Z_{\mu_1\ldots\mu_s} = \gamma^\beta\Gamma^{(1)}_{\beta;\mu_1\ldots\mu_s} \tag{3.3.3}$$

$$= \gamma^\lambda\partial_\lambda\psi_{\mu_1\ldots\mu_s} - \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\gamma^\nu\psi_{\nu\mu_2\ldots\mu_s} = 0 \tag{3.3.4}$$

and here the spinor indices are not written. We define these PDEs (3.3.4) as equations of motion for a massless spin-$(s+\frac{1}{2})$ field. Indeed, when $s = 0$, they reduce to the massless Dirac equations. For a spin-3/2 field, the equations of motion are

$$Z_\mu = \gamma^\nu\partial_\nu\psi_\mu - \partial_\mu\gamma^\nu\psi_\nu \quad , \tag{3.3.5}$$

and they are expressed also as

$$\gamma^\mu\psi_{\mu\nu} = 0 \tag{3.3.6}$$

$$\text{where } \ \psi_{\mu\nu} \equiv \partial_\mu\psi_\nu - \partial_\nu\psi_\mu \quad . \tag{3.3.7}$$

For a spin-$(s+\frac{1}{2})$, $(s \geq 3)$ field, the triple-traceless condition:

$$\gamma^\nu\gamma^\xi\gamma^\rho\psi_{\nu\xi\rho\mu_4\ldots\mu_s} = 0 \tag{3.3.8}$$

with regard to gamma matrices is imposed.

Let us adopt

$$G_{\mu_2\ldots\mu_s} = \gamma^\rho\psi_{\rho\mu_2\ldots\mu_s} - \frac{1}{2s}\sum_{\mu,\text{level}-1}^{\text{sym}} \gamma_{\mu_2}\gamma^\xi\gamma^\rho\psi_{\xi\rho\mu_3\ldots\mu_s} = 0 \tag{3.3.9}$$

as gauge-fixing conditions for a fermionic field. Since $\gamma^{\mu_2} G_{\mu_2 \dots \mu_s = 0}$, we obtain

$$\frac{s-3}{s} \gamma^\alpha \gamma^\beta \psi_{\alpha\beta\mu_3 \dots \mu_s} = 0 \quad , \tag{3.3.10}$$

and therefore, with the similar method which is used in (3.2.12) , the number of the independent components of $G_{\mu_2 \dots \mu_s}$ is $4 \times ({}_{s+2}C_3 - {}_{s+1}C_3) = 2s(s+1)$, in which the number 4 comes from the number of spinor indices. However in case of $s = 3$, (3.3.10) vanishes, and so the number of independent components of $G_{\mu_2\mu_3}$ is $4 \times ({}_5C_3) = 40$ in this[29] case. Moreover, since the gauge parameter $\kappa_{\mu_2 \dots \mu_s}$ is traceless with regard to the gamma matrices, the gauge variations of $G_{\mu_2 \dots \mu_s}$ are expressed as

$$\delta G_{\mu_2 \dots \mu_s} = \gamma^\alpha \partial_\alpha \kappa_{\mu_2 \dots \mu_s} - \frac{1}{s} \sum_{\mu, \text{level}-1}^{\text{sym}} \gamma_{\mu_2} \partial^\alpha \kappa_{\alpha\mu_3 \dots \mu_s} = 0 \quad , \tag{3.3.11}$$

and therefore the gauge conditions (3.3.9) enable us to conduct further regauge transformations with parameters $\kappa_{\mu_2 \dots \mu_s}$ for which (3.3.11) become zero. Since (3.3.11) is gamma-traceless and defines an initial value problem which has no constraints as long as its gamma-traceless property is preserved in time, regauging makes the removal of $2s(s+1)$ degrees of freedom possible.

Next we consider the constraint equations in fermionic cases. The equations of motion (3.3.4) contain constraints

$$Z_{0j_2 \dots j_s} - \gamma_0 \gamma^i Z_{ij_2 \dots j_s} = 0 \tag{3.3.12}$$

where the indices $i, j, k = 1, 2, 3$ are spatial components, and so only spatial derivatives are used in these constraints. The number of these constraints is $4 \times ({}_{s+1}C_2) = 2s(s+1)$; the number is the same as the number $4 \times ({}_{s+2}C_3 - {}_{s+1}C_3)$ of the independent components of the gauge parameter $\kappa_{\mu_2 \dots \mu_s}$ which is gamma-traceless. Unlike bosonic cases, the constrains cannot turn into evolution equations for fermionic cases because the fermionic gauge conditions (3.3.9) have no derivatives.

An example of fermionic constraints: Before we proceed to count the dynamical degrees of freedom for fermionic fields, we show an example of constraint equations. Here a massive Rarita-Schwinger field in vacuum are considered. The mass term is added to massless equations of motion (3.3.4) , and the gauge-fixing conditions (3.3.9) is taken[30]. Accordingly the equations of motion are written as

$$Z_\mu = \gamma^\nu \partial_\nu \psi_\mu + i m_{3/2} \psi_\mu = 0 \tag{3.3.13}$$

[29] Thus the case of $s = 3$ should be treated in a different way in order to coordinate the number of independent components of $G_{\mu\nu}$ in my opinion.

[30] It follows that here the gauge-fixing conditions are $\gamma^\nu \psi_\nu = 0$.

where $m_{3/2}$ is the mass of the spin-3/2 particle. According to (3.3.12) , the constraint equation are

$$Z_0 - \gamma_0 \gamma^k Z_k = 0 \quad , \tag{3.3.14}$$

and after some calculation, the constraints become

$$\gamma^\nu \partial_\nu \psi_0 - \gamma_0 \partial^\nu \psi_\nu + i m_{3/2} \psi_0 = 0 \quad . \tag{3.3.15}$$

When the equations of motion is valid, $Z_0 = 0$, and so we substitute (3.3.13) in (3.3.15) . This leads to

$$\gamma_0 \partial^\nu \psi_\nu = 0 \tag{3.3.16}$$

$$\Leftrightarrow \partial^\nu \psi_\nu = 0 \quad , \tag{3.3.17}$$

that is, the constraints (3.3.17) are obtained.

Dynamical degrees of freedom for massless fermionic fields: Here we compute the dynamical components of a spin-$(s+\frac{1}{2})$ field. Since it is triple traceless with regard to the gamma matrices, the components of the field $\psi_{\mu_1 \dots \mu_s}$ are $4 \times ({}_{s+3}C_3 - {}_sC_3) = 6s^2 + 6s + 4$. Then, from this, the number $2s(s+1)$ of constraints, the number $2s(s+1)$ of gauge conditions and the number $2s(s+1)$ of regauge possibilities should be subtracted, which results in 4 remaining degrees of freedom for all s.

The equations of motion $Z_{\mu_1 \dots \mu_s}$ and the gauge conditions $G_{\mu_2 \dots mu_s}$ give rise to

$$\gamma^\nu \partial_\nu \psi_{\mu_1 \dots \mu_s} - \frac{1}{2s} \sum_{\mu,\mathrm{level}-2}^{\mathrm{sym}} (\gamma_{\mu_1} \partial_{\mu_2} + \gamma_{\mu_2} \partial_{\mu_1}) \gamma^\xi \gamma^\rho \psi_{\xi\rho\mu_3 \dots \mu_s} = 0 \quad , \tag{3.3.18}$$

which is not desirable. Then we use the regauge freedom to cancel the second term in (3.3.18) ; the gauge variation of $\gamma^\xi \gamma^\rho \psi_{\xi\rho\mu_3 \dots \mu_s}$ is

$$\delta \gamma^\alpha \gamma^\beta \psi_{\alpha\beta\mu_3 \dots \mu_s} = 2 \partial^\nu \kappa_{\nu\mu_3 \dots \mu_s} \quad . \tag{3.3.19}$$

Under the condition that the field $\psi_{\mu_1 \dots \mu_s}$ satisfies (3.3.18) , with nonvanishing double trace with regard to the gamma matrices, the regauge transformation with parameters

$$\kappa_{\mu_2 \dots \mu_s}(x) = -\frac{1}{2s} \int d^4 y D(x-y) \gamma^\nu \frac{\partial}{\partial y^\nu} \times \sum_{\mu,\mathrm{level}-1}^{\mathrm{sym}} \gamma_{\mu_2} \gamma^\alpha \gamma^\beta \psi_{\alpha\beta\mu_3 \dots \mu_s}(y) \tag{3.3.20}$$

is conducted, where $D(x-y)$ is a Green's function of the d'Alembertian operator. The gauge parameter in (3.3.20) should be traceless and satisfy the regauge conditions (3.3.11) . Therefore after regauging, the gauge conditions are turned into

$$G_{\mu_2 \dots \mu_s} = \gamma^\nu \psi_{\nu\mu_2\dots\mu_s} = 0 \quad , \tag{3.3.21}$$

which leads to the (regauged) equations of motion:

$$Z_{\mu_1 \dots \mu_s} = \gamma^\nu \partial_\nu \psi_{\mu_1\dots\mu_s} = 0 \quad , \tag{3.3.22}$$

that is, a natural generalisation of massless Dirac equations. We know that there remains further regauge freedom with gamma-traceless gauge parameters such that

$$\gamma^\nu \partial_\nu \kappa_{\mu_2\dots\mu_s} = 0 \quad , \tag{3.3.23}$$

and since one can add any gamma-traceless solution of (3.3.23) to the original gauge parameters (3.3.20) without manipulating their properties, there is no uniqueness in the original gauge parameters.

Such a gamma-traceless gauge choice (3.3.21) of the fermion fields let the fields have $4\times({}_{s+3}C_3 - {}_{s+2}C_3) = 2s^2+6s+4$ degrees of freedom. The (regauged) equations of motion (3.3.22) contain both evolution equations and constraint equations; (3.3.12) is still the constraint equations, combined with (3.3.21) . Thus there are still $2s(s+1)$ constraints. The remaining regauges (3.3.23) have $2s(s-1)$ constraints on the gauge parameters. There are $2s(s+1)$ original regauge degrees of freedom. Then, for the purpose of eliminating $\gamma^\alpha\gamma^\beta\psi_{\alpha\beta\mu_3\dots\mu_s}$ in the gauge condition $G_{\mu_2\dots\mu_s} = 0$, the first regauging spends $2s(s-1)$ degrees of freedom. The number of remaining degrees of freedom is $4s$. Hence the dynamical independent components are

$$\text{dof} = (\text{dof of the gamma-traceless field}) \tag{3.3.24}$$

$$- (\text{the number of constraints}) - (\text{dof of the second regauge transformation}) \tag{3.3.25}$$

$$= (2s^2+6s+4) - (2s(s+1)) - 4s = 4 \tag{3.3.26}$$

in the fermionic field. This result is valid for spin-1/2 case, that is, for Dirac equations.

Let us now use the polarisation vector $\epsilon^\mu_\lambda(k)$ and positive- or negative-helicity solutions $u_{k,\pm\lambda}$ of the Dirac equations to construct their corresponding notion which is applicable to higher-spin fermionic fields; the positive energy (plane wave) solutions may take such a form

$$U^{\mu_1\mu_2\dots\mu_s}(k) = \epsilon^{\mu_1}_\lambda(k)\epsilon^{\mu_2}_\lambda(k)\cdots\epsilon^{\mu_s}_\lambda(k)e^{-ik\cdot x} + \epsilon^{\mu_1}_{-\lambda}(k)\epsilon^{\mu_2}_{-\lambda}(k)\cdots\epsilon^{\mu_s}_{-\lambda}(k)e^{-ik\cdot x} \quad , \tag{3.3.27}$$

which satisfies (3.3.21) and (3.3.22) . The helicity $\pm\left(s+\frac{1}{2}\right)$ are carried. In order to realise the 4 dynamical degrees of freedom, we need to consider the negative energy solutions. Later we obtain their specific form.

Lagrangians for massless fermionic fields: This subsection is finalised by constructing the Lagrangian for a spin-($s+\frac{1}{2}$ field. The Lagrangian is given as

$$\mathcal{L} = \frac{-1}{2}\overline{\psi}_{\mu_1\ldots\mu_s}Z^{\mu_1\ldots\mu_s} + \frac{s}{4}\overline{\psi}_{\beta\mu_2\ldots\mu_s}\gamma^\beta\gamma_\gamma Z^{\gamma\mu_2\ldots\mu_s} + \frac{s(s-1)}{8}\overline{\psi}_{\xi\rho\mu_3\ldots\mu_s}\gamma^\xi\gamma^\rho\gamma_\alpha\gamma_\sigma Z^{\alpha\sigma\mu_3\ldots\mu_s} \quad ; \tag{3.3.28}$$

for example, in a case spin-1/2 , we have

$$\mathcal{L} = \frac{-1}{2}\overline{\psi}\gamma^\nu\partial_\nu\psi \quad , \tag{3.3.29}$$

indeed this is a Lagrangian for a massless Dirac field. The equations of motion

$$Z_{\mu_1\ldots\mu_s} - \frac{1}{2}\sum_{\mu,\text{level}-1}^{\text{sym}}\gamma_{\mu_1}\gamma^\alpha Z_{\alpha\mu_2\ldots\mu_s} - \frac{1}{2}\sum_{\mu,\text{level}-2}^{\text{sym}}\eta_{\mu_1\mu_2}\gamma^\alpha\gamma^\beta Z_{\alpha\beta\mu_3\ldots\mu_s} = 0 \tag{3.3.30}$$

is derived from the Lagrangian (3.3.28) , and after one takes its gamma-trace, one obtain[31] the equations of motion $Z_{\mu_1\ldots\mu_s} = 0$ for the field. The gamma-traceless part of the divergence

$$\partial^\xi Z_{\xi\mu_2\ldots\mu_s} - \frac{1}{2}\gamma^\alpha\partial_\alpha\gamma^\beta Z_{\beta\mu_2\ldots\mu_s} - \frac{1}{2}\sum_{\mu,\text{level}-1}^{\text{sym}}\partial_{\mu_2}\gamma^\alpha\gamma^\beta Z_{\alpha\beta\mu_3\ldots\mu_s} = 0 \tag{3.3.31}$$

may refer to a Bianchi identity, which is related with the gauge invariance of the Lagrangian. As an example of this, for a massless Rarita-Schwinger field, the identity holds: (3.3.31) becomes

$$\begin{aligned}\partial^\mu Z_\mu - \frac{1}{2}\gamma^\alpha\partial_\alpha\gamma^\beta Z_\beta &= \partial^\mu\gamma^\nu\left(\partial_\nu\psi_\mu - \partial_\mu\psi_\nu\right)\\ &\quad - \frac{1}{2}\gamma^\alpha\partial_\alpha\gamma^\nu\gamma^\gamma\left(\partial_\nu\psi_\beta - \partial_\beta\psi_\nu\right) &(3.3.32)\\ &= -\partial^\mu\partial_\mu\gamma^\nu\psi_\nu + \gamma^\alpha\partial_\alpha\gamma^\beta\gamma^\nu\partial_\beta\psi_\nu &(3.3.33)\\ &= -\partial^\mu\partial_\mu\gamma^\nu\psi_\nu + \frac{1}{2}\partial_\alpha\partial_\beta\left(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha\right)\gamma^\nu\psi_\nu = 0 \quad . &(3.3.34)\end{aligned}$$

Indeed, this is an identity.

[31]It may be impossible to obtain the equations of motion directly from the Lagrangian.

3.4 Properties of a symmetrised sum

We used symmetrised sums to construct equations of motion for higher-spin fields. In this subsection, the symmetrised sum is discussed.

When we symmetrise a tensor, we add some tensors to the original tensor, and so there are various ways to do it;

$$\text{some people symmetrise as } \sum_{\text{sym}} A_{\mu\nu} = A_{\mu\nu} + A_{\nu\mu} \tag{3.4.1}$$

$$\text{other people do as } \sum_{\text{sym}} A_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right) + \eta_{\mu\nu}\phi \quad , \tag{3.4.2}$$

where $\eta_{\mu\nu}$ is a rank-2 symmetric tensor and ϕ is a scalar. Thus, unlike the usual sum operation $\sum$, a symmetrised sum

$$\sum_{\text{sym}}$$

does not assure the distributive law, that is

$$\sum_{\text{sym}} (D_{\mu\nu} + E_{\mu\nu}) \neq \sum_{\text{sym}} D_{\mu\nu} + \sum_{\text{sym}} E_{\mu\nu} \tag{3.4.3}$$

appears sometimes. For example if $A_{\mu\nu} = p_\mu q_\nu + p_\nu q_\mu$, then

$$\sum_{\text{sym}} (p_\mu q_\nu + p_\nu q_\mu) = p_\mu q_\nu + p_\nu q_\mu = A_{\mu\nu} \quad , \tag{3.4.4}$$

while

$$\sum_{\text{sym}} p_\mu q_\nu + \sum_{\text{sym}} p_\nu q_\mu = 2(p_\mu q_\nu + p_\nu q_\mu) = 2A_{\mu\nu} \quad , \tag{3.4.5}$$

and so

$$\sum_{\text{sym}} (p_\mu q_\nu + p_\nu q_\mu) \neq \sum_{\text{sym}} p_\mu q_\nu + \sum_{\text{sym}} p_\nu q_\mu \quad . \tag{3.4.6}$$

It follows that we should make a symmetrised sum, which is used here, well-defined. Let $A_{\mu_1 \ldots \mu_{s-1}}$ be a rank-(s-1) totally symmetric tensor. We define the level-1 symmetrised sum of the rank-s tensor $\partial_{\mu_1} A_{\mu_2 \ldots \mu_s}$, $(s \in \mathbf{N})$ with regard to the indices $\{\mu_i\}$ as

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1} A_{\mu_2 \ldots \mu_s} \equiv \partial_{\mu_1} A_{\mu_2 \mu_3 \ldots \mu_s} + \partial_{\mu_2} A_{\mu_1 \mu_3 \ldots \mu_s} + \cdots + \partial_{\mu_s} A_{\mu_1 \ldots \mu_{s-1}} \quad , \tag{3.4.7}$$

that is, there appear s terms in this sum. Next we provide the specific expressions of the symmetrised sum for the first derivative of a completely symmetric rank-(s-1) tensor $A_{\mu_1\ldots\mu_{s-1}}$, in which the derivative gives us a different index λ . We define it as

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} \equiv \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} + \partial_{\mu_1} A_{\lambda\mu_2\ldots\mu_{s-1}} + \partial_{\mu_2} A_{\mu_1\lambda\ldots\mu_{s-1}} + \cdots + \partial_{\mu_{s-2}} A_{\mu_1\ldots\lambda\mu_{s-1}} + \partial_{\mu_{s-1}} A_{\mu_1\ldots\mu_{s-2}\lambda} \quad , \tag{3.4.8}$$

is derived, where s terms appear. Since $A_{\mu_1\mu_2\ldots\mu_{s-1}}$ is completely symmetric, we have

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} = \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} + \partial_{\mu_1} A_{\lambda\mu_2\ldots\mu_{s-1}} + \partial_{\mu_2} A_{\lambda\mu_1\mu_3\ldots\mu_{s-1}} + \cdots + \partial_{\mu_{s-2}} A_{\lambda\mu_1\ldots\mu_{s-3}\mu_{s-1}} + \partial_{\mu_{s-1}} A_{\lambda\mu_1\ldots\mu_{s-2}} \quad , \tag{3.4.9}$$

and accordingly we multiply the both sides by γ^λ and taking sum in terms of the index lambda:

$$\gamma^\lambda \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} = \gamma^\lambda \partial_\lambda A_{\mu_1\mu_2\ldots\mu_{s-1}} + \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1} \gamma^\lambda A_{\lambda\mu_2\ldots\mu_{s-1}} \quad , \tag{3.4.10}$$

because $\gamma^\lambda A_{\lambda\mu_2\ldots\mu_{s-1}}$ is a completely symmetric rank-(s-2) tensor. Therefore the second term of right-hand side in (3.4.10) consists of (s-1) terms. We must be careful of the number of terms in these symmetrised sums. For example,

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2} \tag{3.4.11}$$

consists of three terms while

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2\mu_3\mu_4} \tag{3.4.12}$$

consists of five terms. In addition, for the case of rank-s tensor $\partial_{\mu_1} A_{\lambda\mu_2\ldots\mu_{s-1}}$, we define the level-1 symmetrised sum of it as

$$\sum_{\mu\text{level}-1}^{\text{sym}} \partial_{\mu_1} A_{\lambda\mu_2\ldots\mu_{s-1}} \equiv \partial_{\mu_1} A_{\lambda\mu_2\ldots\mu_{s-1}} + \partial_{\mu_2} A_{\lambda\mu_1\ldots\mu_{s-1}} + \cdots + \partial_{\mu_{s-1}} A_{\lambda\mu_1\mu_2\ldots\mu_{s-2}} \quad , \tag{3.4.13}$$

where there appear (s-1) terms, and so we can rewrite (3.4.9) as

$$\sum_{\mu,\text{level}-1}^{\text{sym}} \partial_\lambda A_{\mu_1\mu_2\dots\mu_{s-1}} = \partial_\lambda A_{\mu_1\dots\mu_{s-1}} + \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1} A_{\lambda\mu_2\dots\mu_{s-1}} \quad . \tag{3.4.14}$$

We are required to consider a more complex case. For a rank-s tensor $\partial_{\mu_s} A_{\lambda_1\lambda_2\mu_3\dots\mu_{s-1}}$, we define the level-1 symmetrised sum as

$$\sum_{\mu\text{level}-1}^{\text{sym}} \partial_{\mu_s} A_{\lambda_1\lambda_2\mu_3\dots\mu_{s-1}} \equiv \partial_{\mu_s} A_{\lambda_1\lambda_2\mu_3\dots\mu_{s-1}} + \partial_{\mu_3} A_{\lambda_1\lambda_2\mu_s\mu_4\dots\mu_{s-1}} + \cdots + \partial_{\mu_{s-1}} A_{\lambda_1\lambda_2\mu_s\mu_3\mu_4\dots\mu_{s-2}} \quad , \tag{3.4.15}$$

where there appear (s-2) terms. Accordingly we can calculate like

$$\sum_{\mu\text{level}-1}^{\text{sym}} \partial_{\lambda_1} A_{\lambda_2\mu_2\mu_3\dots\mu_{s-1}} = \partial_{\lambda_1} A_{\lambda_2\mu_2\mu_3\dots\mu_{s-1}} + \sum_{\mu\text{level}-1}^{\text{sym}} \partial_{\mu_2} A_{\lambda_1\lambda_2\mu_3\dots\mu_{s-1}} \quad . \tag{3.4.16}$$

Now we consider a higher-level symmetrised sum of a tensor. We define the level-2 symmetrised sum of rank-(s+ 1) tensor $\partial_{\mu_1}\partial_{\mu_2} A_{\lambda\mu_3\dots\mu_s}$ with regard to the indices $\{\mu_i\}$ as

$$\sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2} A_{\lambda\mu_3\dots\mu_s} \equiv \frac{1}{2} \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1} \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_2} A_{\lambda\mu_3\dots\mu_s} \tag{3.4.17}$$

$$= \partial_{\mu_1}\partial_{\mu_2} A_{\lambda\mu_3\dots\mu_s} + \partial_{\mu_1}\partial_{\mu_3} A_{\lambda\mu_2\dots\mu_s} + \cdots + \partial_{\mu_{s-1}}\partial_{\mu_s} A_{\lambda\mu_1\mu_2\dots\mu_{s-2}} \quad , \tag{3.4.18}$$

where there appear $\frac{1}{2}s(s-1)$ independent terms with regard to the indices $\{\mu_i\}$. By the discussion similar to (3.4.14) , the definition

$$\sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_2}\partial_{\beta_1} \phi_{\beta_2\beta_3\mu_3\dots\mu_s} \equiv \frac{3}{2} \sum_{\mu\text{level}-1}^{\text{sym}} \partial_{\mu_2}\partial_{\beta_1} \phi_{\beta_2\beta_3\mu_3\dots\mu_s} + \sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_2}\partial_{\mu_3} \phi_{\beta_1\beta_2\beta_3\mu_4\dots\mu_s} \tag{3.4.19}$$

is established.

Symmetrisation and Anti-symmetrisation Before we proceed to generalise the notion of the symmetrised sum, we consider symmetrisation and anti-symmetrisation of a spinorial tensor. Let $T^{\mu_1\mu_2\cdots\mu_q}$ be a completely asymmetric rank-(q, 0) tensor, and symmetrisation of this tensor is realised by

$$T^{(\mu_1\mu_2\dots\mu_q)} = \frac{1}{q!} \{T^{\mu_1\mu_2\dots\mu_q} + T^{\mu_1\dots\mu_q\mu_{q-1}} + \cdots + T^{\mu_q\mu_{q-1}\dots\mu_1}\} \quad , \tag{3.4.20}$$

where the round bracket as the superscript of the tensor means the symmetrisation. For example, for a rank-(2,0) tensor $B^{\mu\nu}$ which is not symmetric, the method

$$B^{(\mu\nu)} = \frac{1}{2}\left(B^{\mu\nu} + B^{\nu\mu}\right) \tag{3.4.21}$$

is its symmetrisation. A symmetric rank-(2,0) tensor $S^{\mu\nu}$ is created with vectors $A_1^\mu,\ A_2^\nu$

$$S^{\mu\nu} = A_1^{(\mu}A_2^{\nu)} = \frac{1}{2}\left(A_1^\mu A_2^\nu + A_1^\nu A_2^\mu\right) \tag{3.4.22}$$

If $S^{\mu\nu}$ is a symmetric rank-2 tensor, then we can create a new completely symmetric rank-3 tensor $S^{\lambda\mu\nu}$ with a vector A_3^λ by

$$S^{\lambda\mu\nu} = A_3^{(\lambda}S^{\mu\nu)} \tag{3.4.23}$$

$$= \frac{1}{3}\left\{A_3^\lambda S^{\mu\nu} + A_3^\mu S^{\lambda\nu} + A_3^\nu S^{\mu\lambda}\right\} \quad , \tag{3.4.24}$$

where by definition the normalisation coefficient is $\frac{1}{3!}$, but since $S^{\mu\nu}$ is symmetric, we have doubly[32] same terms. Accordingly, we obtain the coefficients $\frac{1}{3}$. Next we consider how to construct a completely symmetric rank-(4,0) tensor with a completely symmetric rank-(3,0) tensor $S^{\lambda\mu\nu}$ and a vector A_4^κ , and expectedly it is realised as

$$S^{\kappa\lambda\mu\nu} = A_4^{(\kappa}S^{\lambda\mu\nu)} \tag{3.4.25}$$

$$= \frac{1}{4}\left\{A_4^{(\kappa}S^{\lambda\mu\nu)} + A_4^{(\lambda}S^{\kappa\mu\nu)} + A_4^{(\mu}S^{\lambda\kappa\nu)} + A_4^{(\nu}S^{\lambda\mu\kappa)}\right\} \quad . \tag{3.4.26}$$

The other way to construct the completely symmetric tensor $S^{\kappa\lambda\mu\nu}$ is to use two symmetric rank-(2,0) tensors $S_1^{\kappa\lambda},\ S_2^{\mu\nu}$.

$$S^{\kappa\lambda\mu\nu} = S_1^{(\kappa\lambda}S_2^{\mu\nu)} \tag{3.4.27}$$

$$= \frac{1}{6}\left\{S_1^{(\kappa\lambda}S_2^{\mu\nu)} + S_1^{(\kappa\mu}S_2^{\lambda\nu)} + \cdots + S_1^{(\mu\nu}S_2^{\kappa\lambda)}\right\} \tag{3.4.28}$$

Generally, a completely symmetric rank-(q+1 ,0) tensor is created by a completely symmetric rank-(q,0) tensor $S^{\mu_1\cdots\mu_q}$ and a vector A^ν , and the way is easily proposed.

$$S^{\mu_1\cdots\mu_q\mu_{q+1}} = A^{(\nu}S^{\mu_1\cdots\mu_q)} \tag{3.4.29}$$

$$= \frac{1}{q+1}\left\{A^{(\nu}S^{\mu_1\cdots\mu_q)} + A^{(\mu_1}S^{\nu\cdots\mu_q)} + \cdots + A^{(\mu_q}S^{\nu\mu_1\cdots\mu_{q-1})}\right\} \tag{3.4.30}$$

[32] eg $C^\lambda S^{\mu\nu} + C^\lambda S^{\nu\mu} = 2C^\lambda S^{\mu\nu}$

Similarly, we can anti-symmetrise the tensor $T^{\mu_1\mu_2\ldots\mu_q}$

$$T^{[\mu_1\ldots\mu_q]} = \frac{1}{q!}\delta^{\mu_1\ldots\mu_q}_{\nu_1\ldots\nu_q}T^{\nu_1\ldots\nu_q} \tag{3.4.31}$$

Generalised symmetrised sum: Now we extend the notion of this symmetrised sum, provided that $H_{\kappa\lambda}$ is a symmetric tensor,

$$\sum_{\mu,\text{level}-2}^{\text{sym}} H_{\mu_1\mu_2}\xi_{\lambda\mu_3\ldots\mu_s} \equiv H_{\mu_1\mu_2}\xi_{\lambda\mu_3\ldots\mu_s} + H_{\mu_1\mu_3}\xi_{\lambda\mu_2\mu_4\ldots\mu_s} + H_{\mu_1\mu_4}\xi_{\lambda\mu_3\ldots\mu_s} + \cdots + H_{\mu_{s-1}\mu_s}\xi_{\lambda\mu_3\ldots\mu_{s-2}} \quad , \tag{3.4.32}$$

where ξ is a completely symmetric rank-(s-1) tensor. This definition says that this sum is a sum of $\dfrac{s(s-1)}{2}$ independent index permutations of the $\{\mu_k\}$. Accordingly we can consider a case in which $H_{\mu_1\mu_2\ldots\mu_j}$ is a completely symmetric rank-j tensor

$$\begin{aligned}\sum_{\mu,\text{level}-\text{j}}^{\text{sym}} H_{\mu_1\ldots\mu_j}\xi_{\lambda_1\lambda_2\ldots\lambda_{j-1}\mu_{j+1}\ldots\mu_s} &\equiv H_{\mu_1\ldots\mu_j}\xi_{\lambda_1\lambda_2\ldots\lambda_{j-1}\mu_{j+1}\ldots\mu_s} + H_{\mu_1\ldots\mu_{j-1}\mu_{j+1}}\xi_{\lambda_1\lambda_2\ldots\lambda_{j-1}\mu_j\mu_{j+2}\ldots\mu_s} \\ &+ \ldots + H_{\mu_{s-j+1}\ldots\mu_s}\xi_{\lambda_1\lambda_2\ldots\lambda_{j-1}\mu_1\ldots\mu_{s-j}} \quad ,\end{aligned} \tag{3.4.33}$$

this is a generalisation of the symmetrised sum, where independent ${}_sC_j$ terms appear.

4 Acausal properties of Rarita-Schwinger fields

One of major problems with regard to higher-spin fields is causality violation. In 1961, K. Johnson and E. C. G. Sudarshan[33] showed that the Rarita-Schwinger fields with an external potential behaved in an acausal way in a quantum sense. In this section, we see that a minimally-coupled Rarita-Schwinger field with an external electromagnetic field breaks causality in a classical sense.

4.1 Causality violation of classical Rarita-Schwinger fields

So far we have seen that the PDEs (3.3.4) are the equations of motion for a massless spin-$(s+\frac{1}{2})$ field, and that (3.3.9) are the corresponding gauge conditions. Taking these into consideration, we can regard

$$\left\{\gamma^\nu\partial_\nu + im_{3/2}\right\}\psi_\mu = 0 \tag{4.1.1}$$

[33]K. Johnson, E.C.G. Sudarshan, Ann, Phys. (N.Y.) **13**, 126 (1961)

as the equations of motion for a massive spin-3/2 field. When the field has a minimal coupling with an external electromagnetic field, the equations of motion becomes

$$\left\{\gamma^{[\mu\nu\xi]}D_\mu + im_{3/2}\gamma^{[\nu\xi]}\right\}\psi_\xi = 0 \quad , \tag{4.1.2}$$

$$\text{(C.2.6)} \rightarrow \quad \Leftrightarrow \left\{-i\gamma^5\epsilon^{\nu\alpha\beta\xi}\gamma_\alpha D_\beta + im_{3/2}\gamma^{[\nu\xi]}\right\}\psi_\xi = 0 \tag{4.1.3}$$

where $D_\mu = \partial_\mu - iqA_\mu$ and the gammas are antisymmetrised gamma matrices[34]. One thinks that the equations of motion (4.1.2) contain constraints equations. Time derivatives may not be included in constraints, and so for $\nu = 0$ in (4.1.2) , equations

$$\left(D^k - \gamma^j D_j \gamma^k + im_{3/2}\gamma^k\right)\psi_k = 0 \tag{4.1.4}$$

$$j, k = 1, 2, 3 \tag{4.1.5}$$

are obtained. These are first constraints, but they do neither affect ψ_0 nor link the component to other components $\{\psi_j\}_{j=1,2,3}$. They do not play a role of constraints. Thus we should consider second constraints, and in order to derive them we act γ_ν and D_ν on (4.1.2) and (4.1.3) , respectively. For the former operation, directly we compute[35]:

$$\gamma_\nu\left\{\gamma^{[\mu\nu\xi]}D_\mu + im_{3/2}\gamma^{[\nu\xi]}\right\}\psi_\xi = 0 \tag{4.1.6}$$

$$\Leftrightarrow \left(2\eta^{\mu\xi} - 2\gamma^\mu\gamma^\xi\right)D_\mu\psi_\xi + i3m_{3/2}\gamma^\xi\psi_\xi = 0 \quad . \tag{4.1.7}$$

For the latter calculation, we have

$$D_\nu\left\{-i\gamma^5\epsilon^{\nu\alpha\beta\xi}\gamma_\alpha D_\beta + im_{3/2}(\gamma^\nu\gamma^\xi - \eta^{\xi\nu})\right\}\psi_\xi = 0 \quad , \tag{4.1.8}$$

and here we use a symmetric tensor property[36]:

$$\epsilon^{\nu\alpha\beta\xi}D_\nu D_\beta\psi_\xi = \epsilon^{\nu\alpha\beta\xi}\left\{\partial_\nu\partial_\beta\psi_\xi - q^2A_\nu A_\beta\psi_\xi - 2iqA_{[\nu}\partial_{\beta]}\psi_\xi\right\} + \epsilon^{\nu\alpha\beta\xi}(-iq)\psi_\xi\partial_\nu A_\beta \tag{4.1.9}$$

$$= -iq\epsilon^{\nu\alpha\beta\xi}\psi_\xi\partial_\nu A_\beta \tag{4.1.10}$$

$$= iq\epsilon^{\alpha\nu\beta\xi}\psi_\xi\partial_\nu A_\beta \tag{4.1.11}$$

$$= iq\epsilon^{\alpha\nu\beta\xi}\psi_\xi\frac{1}{2}\left(\partial_\nu A_\beta - \partial_\beta A_\nu\right) \tag{4.1.12}$$

$$= iq\epsilon^{\alpha\nu\beta\xi}\psi_\xi\frac{1}{2}F_{\nu\beta} = iq\psi_\xi\tilde{F}^{\alpha\xi} \quad , \tag{4.1.13}$$

[34]See Appendix Formuale for gamma matrices.

[35]We use (C.2.3) (See Appendix Formulae for gamma matrices).

[36]Let $S^{\mu\nu}$ be a symmetric tensor and $A^{\mu\nu}$ be an antisymmetric tensor. Then $S^{\mu\nu}A_{\mu\nu} = 0$.

where we use $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. Therefore, (4.1.8) becomes

$$im_{3/2}\left(\gamma^\nu D_\nu \gamma^\xi - D^\xi\right)\psi_\xi + \gamma^5 q\gamma_\alpha\psi_\xi\tilde{F}^{\alpha\xi} = 0 \quad . \tag{4.1.14}$$

By using the results (4.1.7) and (4.1.14) , we derive the relations:

$$\gamma^\mu\psi_\mu = \frac{2q}{3m_{3/2}^2}\gamma^5\gamma_\alpha\psi_\xi\tilde{F}^{\alpha\xi} \quad , \tag{4.1.15}$$

and these relations are secondary constraints. In addition, (4.1.14) and (4.1.15) lead to

$$D^\xi\psi_\xi = \left(\gamma^\mu D_\mu - i\frac{3}{2}m_{3/2}\right)\frac{2q}{3m_{3/2}^2}\gamma^5\gamma_\alpha\psi_\xi\tilde{F}^{\alpha\xi} \quad , \tag{4.1.16}$$

that is, the secondary constraints consist of (4.1.15) and (4.1.16) .

Characteristic determinants for minimally-coupled RS equations: Now we use (4.1.15) and (4.1.16) to rewrite the original minimally-coupled Rarita-Schwinger equations (4.1.2) . After relatively easy calculation, we have

$$-\left(\gamma^\mu D_\mu + im_{3/2}\right)\psi^\nu + \left(D^\nu - i\frac{1}{2}m_{3/2}\gamma^\nu\right)\frac{2q}{3m_{3/2}^2}\gamma^5\gamma_\alpha\psi_\xi\tilde{F}^{\alpha\xi} = 0 \quad . \tag{4.1.17}$$

These are equations of motion for a minimally-coupled Rarita-Schwinger field with secondary (substantial) constraints. We proceed to compute their characteristic determinant Q . For the normal vector $\xi_\mu = \partial_\mu\phi$ to a characteristic surface $C : \phi(x) = 0$, we have its characteristic matrix:

$$H(\xi_\mu) = \begin{pmatrix} -\not{\xi} + M^0\xi^0 & M^1\xi^0 & M^2\xi^0 & M^3\xi^0 \\ M^0\xi^1 & -\not{\xi} + M^1\xi^1 & M^2\xi^1 & M^3\xi^1 \\ M^0\xi^2 & M^1\xi^2 & -\not{\xi} + M^2\xi^2 & M^3\xi^2 \\ M^0\xi^3 & M^1\xi^3 & M^2\xi^3 & \not{\xi} + M^3\xi^3 \end{pmatrix} \quad , \tag{4.1.18}$$

where $M^\xi = \frac{2q}{3m_{3/2}^2}\gamma^5\gamma_\alpha\tilde{F}^{\alpha\xi}$. The statement (1.2.10) says that in order to show the acausal behaviour of the Rarita-Schwinger field, we should seek for a spacelike surface for the Rarita-Schwinger field. Let us investigate whether a normal vector $\partial_\mu\phi = \xi_\mu = (n, 0, 0, 0)$ to the characteristic surface exists or not. Under this condition, the characteristic matrix (4.1.18) reduces to

$$H(\xi_\mu) = \begin{pmatrix} -\gamma^0 n + M^0 n & M^1 n & M^2 n & M^3 n \\ 0 & -\gamma^0 n & 0 & 0 \\ 0 & 0 & -\gamma^0 n & 0 \\ 0 & 0 & 0 & -\gamma^0 n \end{pmatrix} \quad . \tag{4.1.19}$$

Basically the dual field strength tensor $\tilde{F}^{\alpha\beta}$ is

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}, \tag{4.1.20}$$

but here we can take it as

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & 0 & 0 \\ -B_y & 0 & 0 & 0 \\ -B_z & 0 & 0 & 0 \end{pmatrix} \tag{4.1.21}$$

without loss of generality. Then we have

$$M^0 = C\gamma^5 \left(\gamma^1 B_x + \gamma^2 B_y + \gamma^3 B_z\right) \tag{4.1.22}$$

$$= C\gamma^5 \boldsymbol{\gamma} \cdot \mathbf{B} \tag{4.1.23}$$

where $C = \dfrac{2q}{3m_{3/2}^2}$. We find that $-\gamma^0 n + M^0 n$ takes a block diagonal[37] form. The characteristic determinant $Q(\xi_\mu)$ becomes

$$Q(\xi_\mu) = \det H(\xi_\mu) \tag{4.1.24}$$

$$= \det(-\gamma^0 n + M^0 n) \times \det(-\gamma^0 n) \times \det(-\gamma^0 n) \times \det(-\gamma^0 n) \tag{4.1.25}$$

$$= \left(n^2 - n^2 C^2 \mathbf{B}^2\right)^2 \left(n^4\right)^3 \tag{4.1.26}$$

$$= n^{16} \left(1 - C^2 \mathbf{B}^2\right)^2 \quad . \tag{4.1.27}$$

Hence if the external magnetic field is

$$\mathbf{B}^2 = \frac{1}{C^2} = \left(\frac{3m_{3/2}^2}{2q}\right)^2 \quad , \tag{4.1.28}$$

then the timelike normal vector exits such that the characteristic determinant vanishes, which means that the spacelike surface exists. We find that a minimally-coupled Rarita-Schwinger field violates causality in a classical sense.

[37]Therefore the calculation of the determinant becomes easier.

4.2 Mechanism of causality violation

It turned out that the Rarita-Schwinger field which coupled with an external electromagnetic field brought about violation of causality at a classical level. We now consider the reason why causality is violated. It seems that the $M^\mu n$ terms in the characteristic matrix (4.1.19) play an important role in making a spacelike characteristic surface of Rarita-Schwinger equations. As is defined before, $M^\xi = \frac{2q}{3m_{3/2}^2}\gamma^5\gamma_\alpha \tilde{F}^{\alpha\xi}$ and this is a part of the secondary constraints: (4.1.15) and (4.1.16) . If the secondary constraints vanish, then the minimally-coupled Rarita-Schwinger equations reduce to

$$(\gamma^\mu D_\mu + im_{3/2})\psi_\nu = 0 \quad , \tag{4.2.1}$$

and the field behaves in a causal way. In other words, if

$$D_\nu \gamma^{[\mu\nu\xi]} D_\mu \psi_\xi = 0 \tag{4.2.2}$$

that is, the massless sector of the equations of motion becomes zero in acting D_ν on them, then the causality recovers.

Such a non-local problem arises for other higher-spin fields. For a bosonic spin-s ($s \geq 4$) field, we have imposed the double-traceless conditions (3.2.11) . This suggests that $Z_{\mu_1\dots\mu_s}$ is ∂^{s-2}-closed , and the generalised Poincaré lemma says that $Z_{\mu_1\dots\mu_s}$ is ∂^3-exact at least locally. Accordingly $Z_{\mu_1\dots\mu_s}$ is expressed as

$$Z_{\mu_1\dots\mu_s} = \sum_{\mu,\text{level}-3}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3} C_{\mu_4\dots\mu_s}(x) \quad , \tag{4.2.3}$$

where the rank-(s-3) tensor $C_{\mu_4\dots\mu_s}$ is called a compensator, and it behaves as

$$\delta C_{\mu_4\dots\mu_s} = 3\zeta^\beta{}_{\beta\mu_4\dots\mu_s} \tag{4.2.4}$$

in a gauge transformation. The behaviour under gauge transformation (4.2.4) implies that the compensator is a symmetric tensor. This transformation property is compatible with (3.1.20) and (3.1.21) .

As an example, let us take a massless spin-5 field. The equations of motion are

$$Z_{\mu_1\dots\mu_5} = \partial^2\phi_{\mu_1\dots\mu_5} - \partial^\beta \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\phi_{\beta\mu_2\dots\mu_5} + \sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\phi^\beta{}_{\beta\mu_3\mu_4\mu_5} \tag{4.2.5}$$

$$= \sum_{\mu,\text{level}-3}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3} C_{\mu_4\mu_5} = 0 \quad . \tag{4.2.6}$$

Then we have

$$\partial_\alpha Z^{\alpha\beta}{}_{\beta\mu_4\mu_5} = \partial^2\partial^2 C_{\mu_4\mu_5} + \dots \quad , \tag{4.2.7}$$

where lower dots refer to the terms with lower-ordered derivatives. Accordingly,

$$C_{\mu_4\mu_5} = W \times \partial_\alpha Z^{\alpha\beta}{}_{\beta\mu_4\mu_5} + \cdots \quad , \tag{4.2.8}$$

is derived, where W is a quantity. The equations of motion becomes

$$Z_{\mu_1\dots\mu_5} = \partial^2\phi_{\mu_1\dots\mu_5} - \partial^\beta \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\phi_{\beta\mu_2\dots\mu_5} + \sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\phi^\beta{}_{\beta\mu_3\mu_4\mu_5} \tag{4.2.9}$$

$$= \sum_{\mu,\text{level}-3}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\left(W \times \partial_\alpha Z^{\alpha\beta}{}_{\beta\mu_4\mu_5} + \cdots\right) = 0 \tag{4.2.10}$$

$$\Leftrightarrow \partial^2\phi_{\mu_1\dots\mu_5} - \partial^\beta \sum_{\mu,\text{level}-1}^{\text{sym}} \partial_{\mu_1}\phi_{\beta\mu_2\dots\mu_5} + \sum_{\mu,\text{level}-2}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\phi^\beta{}_{\beta\mu_3\mu_4\mu_5}$$
$$- \sum_{\mu,\text{level}-3}^{\text{sym}} \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\left(W \times \partial_\alpha Z^{\alpha\beta}{}_{\beta\mu_4\mu_5} + \cdots\right) = 0 \quad . \tag{4.2.11}$$

Violation of causality stems from the last term in (4.2.11) . Similarly the non-local issues arises for fermionic[38] higher-spin fields.

5 Supergravity and higher spin fields

Last section demonstrated that causality broke down in a classical sense for a minimally-coupled Rarita-Schwinger field. This inconsistency in QFT should be avoided, and so supergravity is adopted here. In this section, we see[39] that causality is restored for a classical Rarita-Schwinger field.

5.1 First order formulation for SUGRA

Let us consider a minimally-coupled Rarita-Schwinger field in supergravity. The Lagrangian

$$L = -\frac{eR}{2} - \frac{1}{2}\epsilon^{\lambda\mu\nu\xi}\overline{\psi}_\lambda\gamma^5\gamma_\mu D_\nu\psi_\xi \tag{5.1.1}$$

[38] Here fermionic issues are not treated.

[39] In this section, we set $\kappa^2 = 16\pi G$, and do $\kappa = 1$.

is regarded as the function of gravitino, vielbeins $e_{\nu b}$ and spin-connections $\omega_{\nu,bc}$ in the first order formulation, where the three variables are independent. Here we have

$$e = \mathrm{det} e_{\nu b} \ , \quad R = e^\mu_b e^\nu_c R_{\mu\nu}{}^{bc} \tag{5.1.2}$$

$$[\nabla_\mu, \nabla_\nu] = -\frac{1}{2} R_{\mu\nu bc} \frac{1}{2} \gamma^{[bc]} \quad , \tag{5.1.3}$$

where the covariant derivative ∇_ν on the Rarita-Schwinger field is defined as

$$\nabla_\mu = \partial_\mu - \frac{1}{2} \omega_{\mu,bc} \frac{1}{2} \gamma^{[bc]} \quad . \tag{5.1.4}$$

The equations of motion are

$$Z^\lambda \equiv \epsilon^{\lambda\mu\nu\xi} \left(\gamma_\mu \nabla_\nu - \frac{1}{4} \gamma_\rho C_{\mu\nu}{}^\rho \right) \psi_\xi = 0 \tag{5.1.5}$$

$$C_{\mu\nu}{}^\rho = \frac{1}{2} \overline{\psi}_\mu \gamma^\rho \psi_\nu \tag{5.1.6}$$

$$G^{\rho\mu} = \frac{i}{2} \epsilon^{\lambda\mu\nu\xi} \overline{\psi}_\lambda \gamma^5 \gamma^\rho \nabla_\nu \psi_\xi \quad . \tag{5.1.7}$$

In addition,

$$G^\mu_a = R^\mu_a - \frac{1}{2} e^\mu_a R \tag{5.1.8}$$

is the asymmetric Einstein tensor, where

$$R_{\mu b} = R_{\mu\lambda b}{}^\lambda \quad . \tag{5.1.9}$$

The torsion is

$$C_{\mu\nu}{}^b = \nabla_\mu e^b_\nu - \nabla_\nu e^b_\mu \tag{5.1.10}$$

$$= \partial_\mu e^b_\nu - \partial_\nu e^b_\mu - \omega_\mu{}^a{}_b e^b_\nu + \omega_\nu{}^a{}_b e^b_\mu \quad . \tag{5.1.11}$$

Then, by using (5.1.3) and the cyclic identity

$$\epsilon^{\lambda\mu\nu\xi} \left(R_{\lambda\mu\nu}{}^b - \nabla_\lambda C_{\mu\nu}{}^b \right) = 0 \quad , \tag{5.1.12}$$

we have

$$\nabla_\lambda Z^\lambda = \frac{1}{2} G^{\lambda\mu} \gamma^5 \gamma_\lambda \psi_\mu + \frac{1}{4} \epsilon^{\lambda\mu\nu\xi} C_{\lambda\mu}{}^\rho \gamma_\rho \nabla_\nu \psi_\xi \quad . \tag{5.1.13}$$

The right-hand side in (5.1.13) corresponds to the non-zero terms in (4.1.13) for the electrically-coupled Rarita-Schwinger field. As is stated in (4.2.2) , if these terms become zero, then causality is restored.

The action $S = \int L d^4 x$ is invariant under the local supersymmetry transformation:

$$\delta e^a_\mu = i\overline{\alpha}\gamma^a \psi_\mu \ , \quad \delta\psi_\mu = 2\nabla_\mu \alpha \tag{5.1.14}$$

$$\delta\omega_\mu{}^{ab} = B_\mu{}^{ab} - \frac{1}{2} e^b_\mu B_c{}^{ac} + \frac{1}{2} e^a_\mu B_c{}^{bc} \tag{5.1.15}$$

where

$$B_a{}^{\lambda\mu} = i\epsilon^{\lambda\mu\nu\xi}\overline{\alpha}\gamma^5\gamma_a \nabla_\nu \psi_\xi \quad . \tag{5.1.16}$$

The least action principle states that the variation

$$\delta S = \int \left(\frac{\delta L}{\delta e^b_\nu} e^b_\nu + \frac{\delta S}{\delta\omega_{\mu ab}} \delta\omega_{\mu ab} + \frac{\delta S}{\delta\psi_\mu} \delta\psi_\mu \right) d^4 x \tag{5.1.17}$$

$$= 0 \quad . \tag{5.1.18}$$

By using the transformation (5.1.14) and integrating by parts, we understand that $\nabla_\mu Z^\mu$ should become zero. Therefore, supergravity restores the causality for the minimally-coupled Rarita-Schwinger field.

6 Quantisation of higher-spin fields

In this section quantisation of higher-spin fields are considered. First the notion of a polarisation 4-vector is generalised, and with the new notion, quantisation of a field with an arbitrary spin is conducted. Second the projection operator for a higher-spin field is introduced, and the Feynman propagator for such a higher-spin field is defined with the projection operator.

6.1 Quantisation with $\epsilon_\lambda^{\mu_1 \ldots \mu_s}$, $\epsilon_\lambda^{*\mu_1 \ldots \mu_s}$

This subsection argues the quantisation of higher-spin fields with positive or negative energy wave functions. Firstly Let us consider a bosonic field ϕ of a particle with rest mass m and spin-s. Then we generalise the notion of polarisation vectors $\epsilon_\mu(p)$ in QED, introducing positive/negative energy wave functions $\epsilon_\lambda^{\mu_1\cdots\mu_s}$, $\overline{\epsilon}_\lambda^{\mu_1\cdots\mu_s}$:

$$\epsilon_\lambda^{\mu_1\cdots\mu_s}(p) \equiv \sum_{\lambda_1,\ldots,\lambda_s=-1}^{1} \delta_{\lambda_1+\cdots+\lambda_s,\lambda} \times \sqrt{\frac{2^s(s+\lambda)!(s-\lambda)!}{(2s)!\prod_{i=1}^{s}(1+\lambda_i)!(1-\lambda_i)!}} \prod_{j=1}^{s} \epsilon_{\lambda_j}^{\mu_j}(p) \tag{6.1.1}$$

$$\epsilon_\lambda^{*\mu_1\cdots\mu_s}(p) \equiv \sum_{\lambda_1,\ldots,\lambda_s=-1}^{1} \delta_{\lambda_1+\cdots+\lambda_s,\lambda} \times \sqrt{\frac{2^s(s+\lambda)!(s-\lambda)!}{(2s)!\prod_{i=1}^{s}(1+\lambda_i)!(1-\lambda_i)!}} \prod_{j=1}^{s} \epsilon_{\lambda_j}^{*\mu_j}(p) \quad , \tag{6.1.2}$$

where ϵ_λ^μ, $\epsilon_\lambda^{*\mu}$ are positive and negative energy wave functions for a spin-1 field, and additionally ϵ_λ^μ may be interpreted as the eigenstates[40] of the helicity operator $\mathbf{j}\cdot\frac{\mathbf{p}}{|\mathbf{p}|} = \mathbf{s}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}$ with eigenvalues $\lambda = -1, 0, 1$. The relations between ϵ_λ^μ and $\epsilon_\lambda^{*\mu}$ are

$$\epsilon_\lambda^{*\mu} = (-1)^\lambda \epsilon_{-\lambda}^\mu \quad , \tag{6.1.3}$$

and additionally we also define the functions for spin-0 as

$$\epsilon_0(p) = \epsilon_0^*(p) = 1 \quad . \tag{6.1.4}$$

We realise that $\epsilon_{s_z}^\mu(p)$ is the polarisation vector for a spin-1 field in quantum field theory. According to the definitions of positive/negative energy wave functions, for spin-2 , we have

$$\epsilon_2^{\mu_1\mu_2}(p) = \epsilon_1^{\mu_1}(p)\epsilon_1^{\mu_2}(p) \ , \quad \epsilon_1^{\mu_1\mu_2} = \frac{1}{\sqrt{2}}\left\{\epsilon_1^{\mu_1}\epsilon_0^{\mu_2} + \epsilon_0^{\mu_1}\epsilon_1^{\mu_2}\right\} \tag{6.1.5}$$

$$\epsilon_0^{\mu_1\mu_2} = \frac{1}{\sqrt{6}}\left\{\epsilon_1^{\mu_1}\epsilon_{-1}^{\mu_2} + 2\epsilon_0^{\mu_1}\epsilon_0^{\mu_2} + \epsilon_{-1}^{\mu_1}\epsilon_1^{\mu_2}\right\} \tag{6.1.6}$$

$$\epsilon_{-1}^{\mu_1\mu_2} = \frac{1}{\sqrt{2}}\left\{\epsilon_0^{\mu_1}\epsilon_{-1}^{\mu_2} + \epsilon_{-1}^{\mu_1}\epsilon_0^{\mu_2}\right\} \ , \quad \epsilon_{-2}^{\mu_1\mu_2} = \epsilon_{-1}^{\mu_1}\epsilon_{-1}^{\mu_2} \quad , \tag{6.1.7}$$

and similar formulae are also derived for the negative energy wave functions. Generally, a positive energy wave function has the following properties:

$$\epsilon^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_s}(p) = \epsilon^{\mu_1\cdots\mu_j\cdots\mu_i\cdots\mu_s}(p) \tag{6.1.8}$$

$$p_\nu \epsilon^{\nu\mu_2\cdots\mu_s}(p) = 0 \ , \quad \epsilon^{\nu\nu\mu_3\cdots\mu_s} = 0 \quad , \tag{6.1.9}$$

and same thing is true for a negative energy wave functions. As the normalisation conditions

$$\eta_{\mu_1\nu_1}\cdots\eta_{\mu_s\nu_s}\epsilon_\lambda^{\mu_1\cdots\mu_s}\epsilon_{\lambda'}^{*\nu_1\cdots\nu_s} = -\delta_{\lambda,\lambda'} \tag{6.1.10}$$

[40] In the spinor representation, we have its eigenvalue equations: $\left(\mathbf{s}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\right)\epsilon_\lambda^\mu = \lambda\epsilon_\lambda^\mu$, and each of $\{\epsilon_\lambda^\mu\}_{\mu=0,1,2,3}$ is a vector quantity, while in the momentum representation each of $\{\epsilon_\lambda^\mu\}_{\mu=0,1,2,3}$ is a scalar quantity.

are taken here, and moreover we make an additional assumption that the functions are averaged as

$$\sum_{\lambda,\lambda'=-s}^{s} \epsilon_{\lambda}^{\mu_1\ldots\mu_s}(p)\epsilon_{\lambda'}^{\nu_1\ldots\nu_s}(p)f(\lambda,\lambda) = (-\eta^{\mu_1\nu_1})(-\eta^{\mu_2\nu_2})\cdots(-\eta^{\mu_s\nu_s})\langle f(\lambda,\lambda')\rangle\delta_{\lambda,\lambda'} \quad , \tag{6.1.11}$$

where $f(\lambda,\lambda')$ is a function, and its bracket means an average[41].

By using the energy wave functions, we can express a bosonic field as

$$\phi^{\mu_1\ldots\mu_s}(x) = \sum_{\lambda=-s}^{s}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left\{a_\lambda(\mathbf{p})\epsilon^{\mu_1\ldots\mu_s}(\mathbf{p})e^{-ip\cdot x} + b_\lambda^\dagger\epsilon^{*\mu_1\ldots\mu_s}(\mathbf{p})e^{ip\cdot x}\right\} \quad , \tag{6.1.12}$$

and the field quantisation conditions are

$$[a_\lambda(\mathbf{p}),\ a_{\lambda'}^\dagger(\mathbf{p}')]_- = [b_\lambda(\mathbf{p}),\ b_{\lambda'}^\dagger(\mathbf{p}')]_- = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{\lambda,\lambda'} \tag{6.1.13}$$

with all other commutations being zero, which completes the qunatisation for a field with an integer spin.

Projection operators for bosonic higher-spin fields: Now we define the projection operator for a spin-1 field as

$$P^{\mu;\nu}(1,p) \equiv \eta^{\mu\nu} - \frac{p^\mu p^\nu}{m^2} \quad , \tag{6.1.14}$$

and from (6.1.14) , it follows that

$$P^{\mu;\nu}(1,k) = P^{\nu;\mu}(1,k) \tag{6.1.15}$$

for a spin-1 field. Additionally, its coordinate version is

$$\hat{P}^{\mu;\nu}(1) = \int \frac{d^4}{(2\pi)^4}e^{-ip\cdot x}P^{\mu;\nu}(1,p) = \delta^{(4)}(x)\left\{\delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right\} \quad , \tag{6.1.16}$$

and here we define it as

$$\hat{P}^{\mu;\nu}(1) \equiv \eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2} \quad . \tag{6.1.17}$$

In general cases, for $s = 2n$, $(n \in \mathbf{N})$ the projection operator is expressed as

$$\begin{aligned} P^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s,p) = \left(\frac{1}{s!}\right)^2 \sum_{P(\mu)P(\nu)} \Bigg[&\prod_{i=1}^{s} P^{\mu_i;\nu_i} + C_1 P^{\mu_1;\mu_2}P^{\nu_1\nu_2}\prod_{i=3}^{s}P^{\mu_i;\nu_i} + \cdots \\ &+C_r P^{\mu_1;\mu_2}P^{\nu_1;\nu_2}P^{\mu_3;\mu_4}P^{\nu_3;\nu_4}\ldots P^{\mu_{2r-1};\mu_{2r}}P^{\nu_{2r-1};\nu_{2r}}\prod_{i=2r+1}^{s}P^{\mu_i\nu_i} + \cdots \\ &+C_{\frac{s}{2}}P^{\mu_1;\mu_2}P^{\nu_1;\nu_2}\ldots P^{\mu_{s-1};\mu_s}P^{\nu_{s-1};\nu_s}\Bigg] \quad , \end{aligned} \tag{6.1.18}$$

[41] Here we leave a degree of freedom for the formula (6.1.11) in terms of how to take the average.

where

$$C_r(s) = \left(\frac{-1}{2}\right)^r \frac{s!}{r!(s-2r)!(2s-1)(2s-3)\cdots(2s-2r+1)} \quad , \tag{6.1.19}$$

and $\sum\limits_{P(\mu),P(\nu)}$ refers to the sum in which all the μ's and ν's permute. For $s = 2n+1$, $(n \in \mathbf{N})$, the projection operator

$$\begin{aligned} P^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s,p) = \left(\frac{1}{s!}\right)^2 \sum_{P(\mu)P(\nu)} \Bigg[\prod_{i=1}^{s} P^{\mu_i;\nu_i} + C_1 P^{\mu_1;\mu_2} P^{\nu_1\nu_2} \prod_{i=3}^{s} P^{\mu_i;\nu_i} + \cdots \\ + C_r P^{\mu_1;\mu_2} P^{\nu_1;\nu_2} P^{\mu_3;\mu_4} P^{\nu_3;\nu_4} \ldots P^{\mu_{2r-1};\mu_{2r}} P^{\nu_{2r-1};\nu_{2r}} \prod_{i=2r+1}^{s} P^{\mu_i\nu_i} + \cdots \\ + C_{\frac{(s-1)}{2}} P^{\mu_1;\mu_2} P^{\nu_1;\nu_2} \ldots P^{\mu_{s-2};\mu_{s-1}} P^{\nu_{s-2};\nu_{s-1}} P^{\mu_s\nu_s} \Bigg] \end{aligned} \tag{6.1.20}$$

is written specifically. By definition for a spin-2 field, we have

$$P^{\mu_1\mu_2;\nu_1\nu_2}(2,p) = \frac{1}{4} \sum_{P(\mu),P(\nu)} \left[P^{\mu_1\nu_1}(p) P^{\mu_2\nu_2}(p) - \frac{1}{3} P^{\mu_1\mu_2}(p) P^{\nu_1\nu_2}(p) \right] \tag{6.1.21}$$

$$\begin{aligned} &= \frac{1}{4}\left\{ P^{\mu_1\nu_1}(p)P^{\mu_2\nu_2}(p) - \frac{1}{3}P^{\mu_1\mu_2}(p)P^{\nu_1\nu_2}(p) \right\} + \frac{1}{4}\left\{ P^{\mu_2\nu_1}(p)P^{\mu_1\nu_2}(p) - \frac{1}{3}P^{\mu_2\mu_1}(p)P^{\nu_1\nu_2}(p) \right\} \\ &+ \frac{1}{4}\left\{ P^{\mu_1\nu_2}(p)P^{\mu_2\nu_1}(p) - \frac{1}{3}P^{\mu_1\mu_2}(p)P^{\nu_2\nu_1}(p) \right\} + \frac{1}{4}\left\{ P^{\mu_2\nu_2}(p)P^{\mu_1\nu_1}(p) - \frac{1}{3}P^{\mu_2\mu_1}(p)P^{\nu_2\nu_1}(p) \right\} \end{aligned} \tag{6.1.22}$$

and accordingly

$$P^{\mu_1\mu_2;\nu_1\nu_2}(2,p) = \frac{1}{2}P^{\mu_1;\nu_1}P^{\mu_2;\nu_2} + \frac{1}{2}P^{\mu_1\nu_2}P^{\mu_2\nu_1} - \frac{1}{3}P^{\mu_1\mu_2}P^{\nu_1\nu_2} \quad , \tag{6.1.23}$$

$$\begin{aligned} &= \frac{1}{2}\left\{ \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \frac{p^{\mu_1}p^{\nu_1}}{m^2}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_1}\frac{p^{\mu_2}p^{\nu_2}}{m^2} \right\} + \frac{1}{2}\left\{ \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} - \frac{p^{\mu_1}p^{\nu_2}}{m^2}\eta^{\mu_2\nu_1} - \eta^{\mu_1\nu_2}\frac{p^{\mu_2}p^{\nu_1}}{m^2} \right\} \\ &- \frac{1}{3}\left\{ \eta^{\mu_1\mu_2}\eta^{\nu_2\nu_1} - \frac{p^{\mu_1}p^{\mu_2}}{m^2}\eta^{\nu_2\nu_1} - \eta^{\mu_1\mu_2}\frac{p^{\nu_2}p^{\nu_1}}{m^2} \right\} + \frac{2}{3}\frac{1}{m^4}p^{\mu_1}p^{\mu_2}p^{\nu_1}p^{\nu_2} \end{aligned} \tag{6.1.24}$$

is derived. Correspondingly, we define its coordinate-oriented version as[42]

$$\hat{P}^{\mu_1\mu_2;\nu_1\nu_2}(2) = \frac{1}{4} \sum_{P(\mu),P(\nu)} \left[\hat{P}^{\mu_1;\nu_1}\hat{P}^{\mu_2;\nu_2} - \frac{1}{3}\hat{P}^{\mu_1;\mu_2}\hat{P}^{\nu_1;\nu_2} \right] \quad . \tag{6.1.25}$$

[42] Hereafter we treat projection operators only in the momentum space.

Let us proceed. The projection operator for a spin-3 field is

$$P^{\mu_1\mu_2\mu_3;\nu_1\nu_2\nu_3}(3,p) = \left(\frac{1}{3!}\right)^2 \sum_{P(\mu),P(\nu)} \{P^{\mu_1;\nu_1}P^{\mu_2;\nu_2}P^{\mu_3;\nu_3} + C_1(3)P^{\mu_1;\mu_2}P^{\nu_1;\nu_2}P^{\mu_3;\nu_3}\} \tag{6.1.26}$$

$$C_1(3) = \frac{-3}{5} \quad , \tag{6.1.27}$$

and after relatively tough calculation,

$$\begin{aligned} P^{\mu_1\mu_2\mu_3;\nu_1\nu_2\nu_3}(3,p) = \frac{1}{6}\{&P^{\mu_1;\nu_1}P^{\mu_2;\nu_2}P^{\mu_3;\nu_3} + P^{\mu_1;\nu_1}P^{\mu_2;\nu_3}P^{\mu_3;\nu_2} + P^{\mu_1;\nu_2}P^{\mu_2;\nu_1}P^{\mu_3;\nu_3} + P^{\mu_1;\nu_2}P^{\mu_2;\nu_3}P^{\mu_3;\nu_1} \\ &+P^{\mu_1;\nu_3}P^{\mu_2;\nu_2}P^{\mu_3;\nu_1} + P^{\mu_1;\nu_3}P^{\mu_2;\nu_1}P^{\mu_3;\nu_2}\} \\ &-\frac{1}{15}\{P^{\mu_1;\mu_2}P^{\nu_1;\nu_2}P^{\mu_3;\nu_3} + P^{\mu_1;\mu_3}P^{\nu_1;\nu_2}P^{\mu_2;\nu_3} + P^{\mu_2;\mu_3}P^{\nu_1;\nu_2}P^{\mu_1;\nu_3} \\ &+P^{\mu_1;\mu_2}P^{\nu_1;\nu_3}P^{\mu_3;\nu_2} + P^{\mu_1;\mu_3}P^{\nu_1;\nu_3}P^{\mu_2;\nu_2} + P^{\mu_2;\mu_3}P^{\nu_1;\nu_3}P^{\mu_1;\nu_2} \\ &+P^{\mu_1;\mu_2}P^{\nu_2;\nu_3}P^{\mu_3;\nu_1} + P^{\mu_1;\mu_3}P^{\nu_2;\nu_3}P^{\mu_2;\nu_1} + P^{\mu_2;\mu_3}P^{\nu_2;\nu_3}P^{\mu_1;\nu_1}\} \end{aligned} \tag{6.1.28}$$

is derived.

(6.1.8) and (6.1.9) indicate that a projection operator has the following properties:

$$P^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_i\ldots\nu_j\ldots\nu_s}(s,p) = P^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_j\ldots\nu_i\ldots\nu_s}(s,p) \tag{6.1.29}$$

$$p_\nu P^{\mu_1\ldots\mu_s;\nu\nu_2\ldots\nu_s}(s,p) = 0 \ , \quad P^{\mu_1\ldots\mu_s;\nu\nu\nu_3\ldots\nu_s}(s,p) = 0 \tag{6.1.30}$$

$$P^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s,p)P^{\nu_1\ldots\nu_s;\xi_1\ldots\xi_s}(s,p) = P^{\mu_1\ldots mu_s;\xi_1\ldots\xi_s}(s,p) \tag{6.1.31}$$

Quantisation of fermion fields with positive/negative energy wave functions: Next we consider fields of particles with spin-$(s+\frac{1}{2})$, where s is an integer. The positive and negative energy wave functions for fermion fields are

$$U^{\mu_1\ldots\mu_s}_{s_z}(p) = \sqrt{\frac{s+\frac{1}{2}+s_z}{2s+1}}\epsilon^{\mu_1\ldots\mu_s}_{s_z-\frac{1}{2}}(p)u_{\frac{1}{2}}(p) + \sqrt{\frac{s+\frac{1}{2}-s_z}{2s+1}}\epsilon^{\mu_1\ldots\mu_s}_{s_z+\frac{1}{2}}(p)u_{\frac{-1}{2}}(p) \tag{6.1.32}$$

$$V^{\mu_1\ldots\mu_s}_{s_z}(p) = \sqrt{\frac{s+\frac{1}{2}+s_z}{2s+1}}\epsilon^{*\mu_1\ldots\mu_s}_{s_z-\frac{1}{2}}(p)v_{\frac{1}{2}}(p) + \sqrt{\frac{s+\frac{1}{2}-s_z}{2s+1}}\epsilon^{*\mu_1\ldots\mu_s}_{s_z+\frac{1}{2}}(p)v_{\frac{-1}{2}}(p) \quad , \tag{6.1.33}$$

where u_r, v_r $(r=\pm\frac{1}{2})$ are the plane wave solutions of the Dirac equation for a particle and an anti-particle, respectively. With (6.1.4) ,

$$U_\lambda(p) = u_{p\lambda} \ , \quad V_\lambda(p) = v_{p\lambda} \tag{6.1.34}$$

are obtained for a spin-1/2 field. For a field with spin-3/2 , we have

$$U^{\mu}_{\frac{3}{2}}(p) = \epsilon^{\mu}_{1}(p) u_{p\frac{1}{2}} \ , \quad U^{\mu}_{\frac{1}{2}}(p) = \sqrt{\frac{2}{3}} \epsilon^{\mu}_{0}(p) u_{p\frac{1}{2}} + \sqrt{\frac{1}{3}} \epsilon^{\mu}_{1}(p) u_{p\frac{-1}{2}} \tag{6.1.35}$$

$$U^{\mu}_{-\frac{1}{2}}(p) = \sqrt{\frac{1}{3}} \epsilon^{\mu}_{-1}(p) u_{p\frac{1}{2}} + \sqrt{\frac{2}{3}} \epsilon^{\mu}_{0}(p) u_{p\frac{-1}{2}} \ , \quad U^{\mu}_{-\frac{3}{2}}(p) = \epsilon^{\mu}_{-1}(p) u_{p\frac{-1}{2}} \tag{6.1.36}$$

and

$$V^{\mu}_{\frac{3}{2}}(p) = \epsilon^{*\mu}_{1}(p) v_{p\frac{1}{2}} \ , \quad V^{\mu}_{\frac{1}{2}}(p) = \sqrt{\frac{2}{3}} \epsilon^{*\mu}_{0}(p) v_{p\frac{1}{2}} + \sqrt{\frac{1}{3}} \epsilon^{*\mu}_{1}(p) v_{p\frac{-1}{2}} \tag{6.1.37}$$

$$V^{\mu}_{-\frac{1}{2}}(p) = \sqrt{\frac{1}{3}} \epsilon^{*\mu}_{-1}(p) v_{p\frac{1}{2}} + \sqrt{\frac{2}{3}} \epsilon^{*\mu}_{0}(p) v_{p\frac{-1}{2}} \ , \quad V^{\mu}_{-\frac{3}{2}}(p) = \epsilon^{*\mu}_{-1}(p) v_{p\frac{-1}{2}} \quad . \tag{6.1.38}$$

Furthermore, for a spin-5/2 field,

$$U^{\mu\nu}_{\frac{5}{2}}(p) = \epsilon^{\mu}_{1} \epsilon^{\nu}_{1} u_{p\frac{1}{2}} \ , \quad U^{\mu\nu}_{\frac{3}{2}} = \sqrt{\frac{2}{5}} \left\{ \epsilon^{\mu}_{1} \epsilon^{\nu}_{0} + \epsilon^{\mu}_{0} \epsilon^{\nu}_{1} \right\} u_{p\frac{1}{2}} + \sqrt{\frac{1}{5}} \epsilon^{\mu}_{1} \epsilon^{\nu}_{1} u_{p\frac{-1}{2}} \tag{6.1.39}$$

$$U^{\mu\nu}_{\frac{1}{2}}(p) = \sqrt{\frac{1}{10}} \left\{ \epsilon^{\mu}_{1} \epsilon^{\nu}_{-1} + 2 \epsilon^{\mu}_{0} \epsilon^{\nu}_{0} + \epsilon^{\mu}_{-1} \epsilon^{\nu}_{1} \right\} u_{p\frac{1}{2}} + \sqrt{\frac{1}{5}} \left\{ \epsilon^{\mu}_{1} \epsilon^{\nu}_{0} + \epsilon^{\mu}_{0} \epsilon^{\nu}_{1} \right\} u_{p\frac{-1}{2}} \tag{6.1.40}$$

$$U^{\mu\nu}_{-\frac{1}{2}}(p) = \sqrt{\frac{1}{5}} \left\{ \epsilon^{\mu}_{0} \epsilon^{\nu}_{-1} + \epsilon^{\mu}_{-1} \epsilon^{\nu}_{0} \right\} u_{p\frac{1}{2}} + \sqrt{\frac{1}{10}} \left\{ \epsilon^{\mu}_{1} \epsilon^{\nu}_{-1} + 2 \epsilon^{\mu}_{0} \epsilon^{\nu}_{0} + \epsilon^{\mu}_{-1} \epsilon^{\nu}_{1} \right\} u_{p\frac{-1}{2}} \tag{6.1.41}$$

$$U^{\mu\nu}_{-\frac{3}{2}}(p) = \sqrt{\frac{1}{5}} \epsilon^{\mu}_{-1} \epsilon^{\nu}_{-1} u_{p\frac{1}{2}} + \sqrt{\frac{2}{5}} \left(\epsilon^{\mu}_{-1} \epsilon^{\nu}_{0} + \epsilon^{\mu}_{0} \epsilon^{\nu}_{-1} \right) u_{p\frac{-1}{2}} \ , \quad U^{\mu\nu}_{-\frac{5}{2}}(p) = \epsilon^{\mu}_{-1} \epsilon^{\nu}_{-1} u_{p\frac{-1}{2}} \tag{6.1.42}$$

and

$$V^{\mu\nu}_{\frac{5}{2}}(p) = \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{1} v_{p\frac{1}{2}} \ , \quad V^{\mu\nu}_{\frac{3}{2}} = \sqrt{\frac{2}{5}} \left\{ \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{0} + \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{1} \right\} v_{p\frac{1}{2}} + \sqrt{\frac{1}{5}} \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{1} v_{p\frac{-1}{2}} \tag{6.1.43}$$

$$V^{\mu\nu}_{\frac{1}{2}}(p) = \sqrt{\frac{1}{10}} \left\{ \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{-1} + 2 \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{0} + \epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{1} \right\} v_{p\frac{1}{2}} + \sqrt{\frac{1}{5}} \left\{ \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{0} + \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{1} \right\} v_{p\frac{-1}{2}} \tag{6.1.44}$$

$$V^{\mu\nu}_{-\frac{1}{2}}(p) = \sqrt{\frac{1}{5}} \left\{ \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{-1} + \epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{0} \right\} v_{p\frac{1}{2}} + \sqrt{\frac{1}{10}} \left\{ \epsilon^{*\mu}_{1} \epsilon^{*\nu}_{-1} + 2 \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{0} + \epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{1} \right\} v_{p\frac{-1}{2}} \tag{6.1.45}$$

$$V^{\mu\nu}_{-\frac{3}{2}}(p) = \sqrt{\frac{1}{5}} \epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{-1} v_{p\frac{1}{2}} + \sqrt{\frac{2}{5}} \left(\epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{0} + \epsilon^{*\mu}_{0} \epsilon^{*\nu}_{-1} \right) v_{p\frac{-1}{2}} \ , \quad V^{\mu\nu}_{-\frac{5}{2}}(p) = \epsilon^{*\mu}_{-1} \epsilon^{*\nu}_{-1} v_{p\frac{-1}{2}} \tag{6.1.46}$$

are written explicitly.

We have their normalisation conditions:

$$\overline{U}^{\mu_1 \ldots \mu_s}_{s_z}(p) U^{\mu_1 \ldots \mu_s}_{s'_z}(p) = 2m \delta_{s_z, s'_z} \tag{6.1.47}$$

$$\overline{V}^{\mu_1 \ldots \mu_s}_{s_z}(p) V^{\mu_1 \ldots \mu_s}_{s'_z}(p) = -2m \delta_{-s_z, -s'_z} \quad , \tag{6.1.48}$$

where

$$\overline{U}^{\mu_1\ldots\mu_s}_{s_z}(p) = \sqrt{\frac{s+\frac{1}{2}+s_z}{2s+1}}\epsilon^{*\mu_1\ldots\mu_s}_{s_z-\frac{1}{2}}(p)\overline{u}_{\frac{1}{2}}(p) + \sqrt{\frac{s+\frac{1}{2}-s_z}{2s+1}}\epsilon^{*\mu_1\ldots\mu_s}_{s_z+\frac{1}{2}}(p)\overline{u}_{\frac{-1}{2}}(p) \tag{6.1.49}$$

$$\overline{V}^{\mu_1\ldots\mu_s}_{s_z}(p) = \sqrt{\frac{s+\frac{1}{2}+s_z}{2s+1}}\epsilon^{\mu_1\ldots\mu_s}_{s_z-\frac{1}{2}}(p)\overline{v}_{\frac{1}{2}}(p) + \sqrt{\frac{s+\frac{1}{2}-s_z}{2s+1}}\epsilon^{\mu_1\ldots\mu_s}_{s_z+\frac{1}{2}}(p)\overline{v}_{\frac{-1}{2}}(p) \tag{6.1.50}$$

$$\overline{u}_r = u_r^\dagger\gamma^0 \quad \overline{v}_r = v_r^\dagger\gamma^0 \quad , \tag{6.1.51}$$

which is consistent with (B.2.11) and (B.2.13) . The fermion field $\psi^{\mu_1\ldots\mu_s}$ is quantised as

$$\psi^{\mu_1\ldots\mu_s}(x) = \sum_{s_z=-(s+\frac{1}{2})}^{s+\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left\{a_{s_z}(p)U^{\mu_1\ldots\mu_s}_{s_z}(p)e^{-ip\cdot x} + b^\dagger_{s_z}(p)V^{\mu_1\ldots\mu_s}_{s_z}(p)e^{ip\cdot x}\right\} \quad , \tag{6.1.52}$$

with the quantisation conditions:

$$[a_\lambda(p), a^\dagger_{\lambda'}(q)]_+ = [b_\lambda(p), b^\dagger_{\lambda'}(q)]_+ = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\lambda\lambda'} \tag{6.1.53}$$

with all other anti-commutators being zero.

Some calculation: For later convenience, we also compute some quantities which are involved in scattering amplitude. As one exercise, for a massive spin-3/2 particle, we calculate the following quantity:

$$\sum_{\lambda=-\frac{3}{2}}^{\frac{3}{2}} U^\mu_\lambda\overline{U}^\alpha_\lambda = \sum_{\lambda=-\frac{3}{2}}^{\frac{3}{2}}\left(\sqrt{\frac{3/2+\lambda}{3}}\epsilon^\mu_{\lambda-\frac{1}{2}}u_{p\frac{1}{2}} + \sqrt{\frac{3/2-\lambda}{3}}\epsilon^\mu_{\lambda+\frac{1}{2}}u_{p\frac{-1}{2}}\right)\left(\sqrt{\frac{3/2+\lambda}{3}}\epsilon^{*\alpha}_{\lambda-\frac{1}{2}}\overline{u}_{p\frac{1}{2}} + \sqrt{\frac{3/2-\lambda}{3}}\epsilon^{*\alpha}_{\lambda+\frac{1}{2}}\overline{u}_{p\frac{-1}{2}}\right) \tag{6.1.54}$$

and it follows from (6.1.11) that[43]

$$\sum_{\lambda=-\frac{3}{2}}^{\frac{3}{2}} U^\mu_\lambda\overline{U}^\alpha_\lambda = \sum_{\lambda=-\frac{3}{2}}^{\frac{3}{2}}\left\{\left(\frac{1}{2}+\frac{\lambda}{3}\right)\epsilon^\mu_{\lambda-\frac{1}{2}}\epsilon^{*\alpha}_{\lambda-\frac{1}{2}}u_{p\frac{1}{2}}\overline{u}_{p\frac{1}{2}} + \left(\frac{1}{2}-\frac{\lambda}{3}\right)\epsilon^\mu_{\lambda+\frac{1}{2}}\epsilon^{*\alpha}_{\lambda+\frac{1}{2}}u_{p\frac{-1}{2}}\overline{u}_{p\frac{-1}{2}}\right\} \tag{6.1.55}$$

$$= \frac{1}{2}(-\eta^{\mu\alpha})u_{p\frac{1}{2}}\overline{u}_{p\frac{1}{2}} + \frac{1}{2}(-\eta^{\mu\alpha})u_{p\frac{-1}{2}}\overline{u}_{p\frac{-1}{2}} \tag{6.1.56}$$

$$= -\frac{1}{2}\eta^{\mu\alpha}(\not{p}+m_{\frac{3}{2}}) \quad , \tag{6.1.57}$$

[43]Now the way of the averaging is that $\langle\frac{3/2+\lambda}{3}\rangle = \frac{1}{4}\sum_{\lambda=-3/2}^{3/2}\frac{3/2+\lambda}{3} = \frac{1}{2}$.

where $m_{\frac{3}{2}}$ is the mass of the spin-3/2 particle. Similarly, for a massive spin-5/2 particle we have

$$\sum_{\lambda=-5/2}^{5/2} U_\lambda^{\mu\nu}(p)\overline{U}_\lambda^{\alpha\beta}(p) = \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\not{p} + m_{\frac{5}{2}}\right) \quad , \tag{6.1.58}$$

where $m_{\frac{5}{2}}$ is the mass of the spin-5/2 particle.

Fermionic projection operators: Now we construct projection operators for fields with an arbitrary half-integer spin. The definition of a projection operator for spin-$s + \frac{1}{2}$ is

$$P_+^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s+\frac{1}{2},p) \equiv \Lambda_+ Q^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s+\frac{1}{2},p) \equiv \frac{s+1}{2s+3}\Lambda_+\gamma_\mu P^{\mu\mu_1\ldots\mu_s;\nu\nu_1\ldots\nu_s}(s+1,p)\gamma_\nu \tag{6.1.59}$$

$$P_-^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s+\frac{1}{2},p) \equiv \Lambda_- Q^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}(s+\frac{1}{2},p) \equiv \frac{s+1}{2s+3}\Lambda_-\gamma_\mu P^{\mu\mu_1\ldots\mu_s;\nu\nu_1\ldots\nu_s}(s+1,p)\gamma_\nu \quad , \tag{6.1.60}$$

and at the same time the projection operator for a spin-$\frac{1}{2}$ field is given by

$$P_+(\frac{1}{2},p) = \Lambda_+ = \sum_{s_z=-\frac{1}{2}}^{\frac{1}{2}} U_{s_z}(p)\overline{U}_{s_z}(p) = \sum_{\lambda=-\frac{1}{2}}^{\frac{1}{2}} u_{p\lambda}\overline{u}_{p\lambda} = m + p_\mu\gamma^\mu = \not{p} + m \tag{6.1.61}$$

$$P_-(\frac{1}{2},p) = \Lambda_- = \sum_{s_z=-\frac{1}{2}}^{\frac{1}{2}} V_{s_z}\overline{V}_{s_z} = \sum_{\lambda=-\frac{1}{2}}^{\frac{1}{2}} v_{p\lambda}\overline{v}_{p\lambda} = -m + \gamma^\mu p_\mu = \not{p} - m \quad , \tag{6.1.62}$$

and accordingly $Q(\frac{1}{2},p) = 1$. For a spin-3/2 field, after some calculations, we have

$$\gamma_\mu P^{\mu\mu_1;\nu\nu_1}(2,p)\gamma_\nu = \frac{5}{2}\eta^{\mu_1\nu_1} - \frac{5}{6}\gamma^{\mu_1}\gamma^{\nu_1} - \frac{1}{6m}\left(p^{\mu_1}\gamma^{\nu_1} + \gamma^{\mu_1}p^{\nu_1}\right) - \frac{4}{3}\frac{p^{\mu_1}p^{\nu_1}}{m^2} \quad , \tag{6.1.63}$$

and accordingly the projection operators

$$P_+^{\mu_1;\nu_1}(\frac{3}{2},p) = \Lambda_+\left(\eta^{\mu_1\nu_1} - \frac{1}{3}\gamma^{\mu_1}\gamma^{\nu_1} - \frac{1}{15m}\left(p^{\mu_1}\gamma^{\nu_1} + \gamma^{\mu_1}p^{\nu_1}\right) - \frac{8}{15}\frac{p^{\mu_1}p^{\nu_1}}{m^2}\right) = \Lambda_+ Q^{\mu_1;\nu_1}(\frac{3}{2},p) \tag{6.1.64}$$

$$P_-^{\mu_1;\nu_1}(\frac{3}{2},p) = \Lambda_-\left(\eta^{\mu_1\nu_1} - \frac{1}{3}\gamma^{\mu_1}\gamma^{\nu_1} - \frac{1}{15m}\left(p^{\mu_1}\gamma^{\nu_1} + \gamma^{\mu_1}p^{\nu_1}\right) - \frac{8}{15}\frac{p^{\mu_1}p^{\nu_1}}{m^2}\right) = \Lambda_- Q^{\mu_1;\nu_1}(\frac{3}{2},p) \tag{6.1.65}$$

$$Q^{\mu_1;\nu_1}(\frac{3}{2},p) = \left(\eta^{\mu_1\nu_1} - \frac{1}{3}\gamma^{\mu_1}\gamma^{\nu_1} - \frac{1}{15m}\left(p^{\mu_1}\gamma^{\nu_1} + \gamma^{\mu_1}p^{\nu_1}\right) - \frac{8}{15}\frac{p^{\mu_1}p^{\nu_1}}{m^2}\right) \tag{6.1.66}$$

are derived.

6.2 Feynman propagator for higher-spin fields

In the previous subsection, higher-spin fields were quantised, and relating projection operators were introduced. Let us now construct Feynman propagators for these fields. Starting by bosonic fields, we treat fermionic case.

Feynman propagator for arbitrary bosonic fields: Here we express the propagator for fields with an arbitrary spin. For a spin-s field, the specific expression of its Feynman propagator $D_F^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}$ is speculated as

$$D_F^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s} = P^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s,p)\frac{-i}{p^2 - m^2} + K^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s,p) \tag{6.2.1}$$

in momentum representation, where $K^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s,p)$ may be some functions of the momentum. Here we define the Feynman propagator as

$$D_F^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s} \equiv P^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s,p)\frac{-i}{p^2 - m^2} \quad , \tag{6.2.2}$$

for a spin-s field in momentum representation for convenience. Indeed, in the case of spin-one, we have

$$D_F^{\mu;\nu} = P^{\mu;\nu}\frac{-i}{p^2 - m^2} = \frac{-i}{p^2 - m^2}\left\{\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right\} \tag{6.2.3}$$

with the aid of (6.1.14) .

Fermionic Feynman propagators: Let us define the propagator for a spin-$(s+\frac{1}{2})$ field as

$$S_F^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s+\frac{1}{2},p) \equiv \frac{i}{\not p - m}Q^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s+\frac{1}{2},p) = \frac{i(\not p + m)}{p^2 - m^2}Q^{\mu_1 \ldots \mu_s;\nu_1 \ldots \nu_s}(s+\frac{1}{2},p) \tag{6.2.4}$$

in momentum space. Since $Q(\frac{1}{2},p) = 1$, this definition leads to

$$S_F(\frac{1}{2},p) = \frac{i(\not p + m)}{p^2 - m^2} \tag{6.2.5}$$

for a Dirac field. For a Rarita-Schwinger field, the Feynman propagator

$$S_F^{\mu_1;\nu_1} = \frac{i}{\not p - m}\left\{\eta^{\mu_1\nu_1} - \frac{1}{3}\gamma^{\mu_1}\gamma^{\nu_1} - \frac{1}{15m}(p^{\mu_1}\gamma^{\nu_1} + \gamma^{\mu_1}p^{\nu_1}) - \frac{8}{15}\frac{p^{\mu_1}p^{\nu_1}}{m^2}\right\} \tag{6.2.6}$$

is obtained.

7 Renormalizability of higher-spin fields

In this section, we see that a spin-2 field coupled with an external electromagnetic field is non-renormalizable.

7.1 Quantised Einstein-Maxwell system

Let us consider a system in which a gravitational field couples with an electromaganetic field. The Lagrangian is

$$L_{GE} = -\sqrt{-\overline{g}}\left\{\frac{R(\overline{g})}{\kappa^2} + \frac{1}{16\pi}\overline{F}_{\mu\nu}\overline{F}_{\alpha\beta}\overline{g}^{\mu\alpha}\overline{g}^{\nu\beta}\right\} \tag{7.1.1}$$

where $\kappa^2 = 16\pi G$ and the fields $\overline{g_{\mu\nu}},\ \overline{F}_{\mu\nu}$ refer to the sums of quantum fields $\kappa h_{mu\nu}, f_{\mu\nu}$ and background fields $g_{\mu\nu}, F_{\mu\nu}$. We consider a term L_B which may break the gauge symmetry:

$$L_{break} = -\frac{1}{8\pi}\sqrt{-g}\left\{(D^\nu h_{\mu\nu} - \frac{1}{2}D_\mu h^\xi_{\ \xi})(D^\nu h^\mu_{\ \nu} - \frac{1}{2}D^\mu h^\xi_{\ \xi}) + (D^\mu A_\mu)^2\right\} \quad , \tag{7.1.2}$$

where de Donger gauge and Lorentz gauge are taken. All operators in the L_{break} are with respect to the background metric $g_{\mu\nu}$. The gauge-fixings lead to the appearance of a new term (Lagrangian) for a vector and a scalr ghost (ξ_μ, ρ) :

$$L_{ghost} = \frac{1}{4\pi}\sqrt{-g}\left\{\xi^{*\mu}(g_{\mu\nu}D_\xi D^\xi - R_{\mu\nu})\xi^\nu + \rho^* D_\xi D^\xi \rho\right\} \quad , \tag{7.1.3}$$

where since the electromagnetic sector of the gauge-breaking Lagrangian L_{break} depends on the metric, the vector ghost exists in (7.1.3) .

The dimensionless[44] parameter ϵ is used in dimensional regularization here. Then the Lagrangian $L_{counter}$ for counter terms of $L_{GE} + L_{break} + L_{ghost}$ is

$$\begin{aligned}
4\pi \times L_{counter} = \frac{\sqrt{-g}}{\epsilon}&\left\{c_1 R_{\mu\nu}R^{\nu\mu} + c_2 R^2 + c_3 R^\mu_{\ \nu\xi\rho}R_\mu^{\ \nu\xi\rho}\right.\\
&\left.+c_4\left(tr(F_{\mu\alpha}F^{\alpha\beta}F_{\beta\nu}F^{\nu\mu}) - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})^2\right)\right\}\\
&+ c_5(F_{\mu\nu}F^{\mu\nu})^2 + c_6 R^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} + c_7 R^{\mu\nu}(F_{\mu\xi}F_\nu^{\ \xi} - \frac{1}{4}g_{\mu\nu}F_{\mu\nu}F^{\mu\nu})\\
&+ c_8 R F_{\mu\nu}F^{\mu\nu} + c_9(D^\mu F_{\mu\nu})(D^\xi F_{\xi\nu}) \quad ,
\end{aligned} \tag{7.1.4}$$

[44] Renormalization of coupling constant is not conducted here, since we have no divergence proportional either to L_{GE} itself or to a cosmological term $\frac{\sqrt{-g}}{\kappa^4}$ in the system. Only κ is the dimensional constant there.

and the identity

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = (4R_{\mu\nu}R^{\mu\nu} - R^2) + (\text{a divergence}) \tag{7.1.5}$$

makes c_3 term be cancelled. $c_5 = c_6 = c_8 = 0$ may be shown by some calculation, and the remaining coefficients are

$$c_1 = \frac{9}{20}\ , \quad c_2 = -\frac{29}{10} \tag{7.1.6}$$

$$c_4 = \frac{13}{24}\ , \quad c_7 = c_9 = \frac{1}{6} \quad . \tag{7.1.7}$$

Field equations obeyed by the unperturbed background fields are used.

$$R_{\mu\nu} = -\frac{1}{2}T_{\mu\nu}\ , \quad D^\mu F_{\mu\nu} = 0 \quad , \tag{7.1.8}$$

and the divergences are not renormalizable unless $L_{counter}$ vanishes. We see that only one term is effective because field equations $R_{\mu\nu}R^{\mu\nu}$,$T_{\mu\nu}T^{\mu\nu}$,$T^{\mu\nu}R_{\mu\nu}$ are equivalent, and $R = 0$. It turns out that the Lagrangian for counter terms does not vanish:

$$4\pi \times L_{counter} = \frac{\sqrt{-g}}{\epsilon}\left(\frac{137}{60}\right)R_{\mu\nu}R^{\mu\nu} \quad , \tag{7.1.9}$$

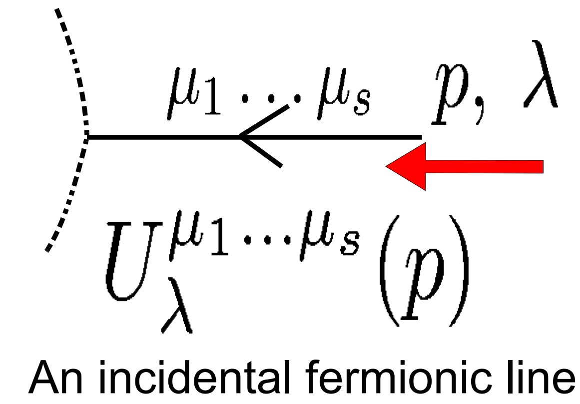
and this means that the theory is not renormalizable at one-loop level.

8 Diagram descriptions of particles with higher-spin

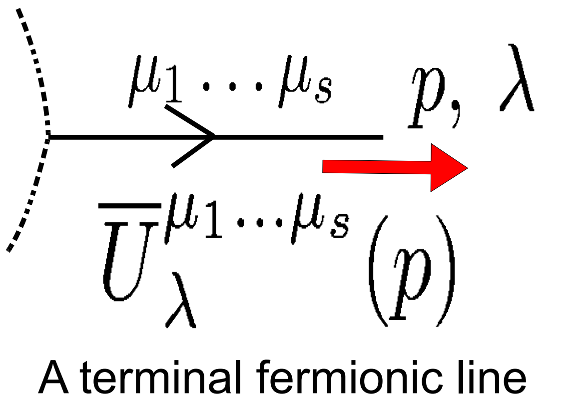
So far we have seen quantisation and relating problems concerning higher-spin fields, and that supergravity makes an solution to the problem of causality violation for a minimally-coupled Rarita-Schwinger field at a classical level. Now we consider kinematics of these fields. In this section, we construct the Feynman rules for these fields.

8.1 Feynman rules for higher-spin fields

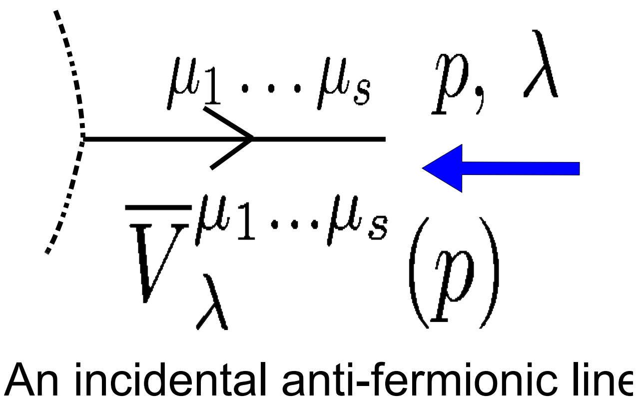
Feynman rules are given for higher-spin fields in this subsection. We treat the rules only in the momentum space. External lines for fermionic or antifermionic waves with momentum p and helicity λ are shown in Figure 15 . Feynman propagators are drawn in Figure 16 , where the incidental point of the virtual



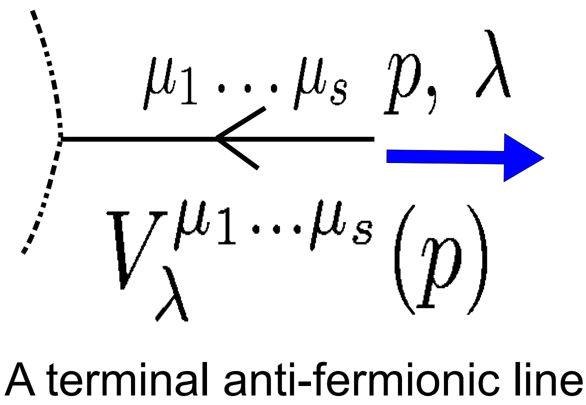
(a) An incidental fermionic line



(b) A terminal fermionic line

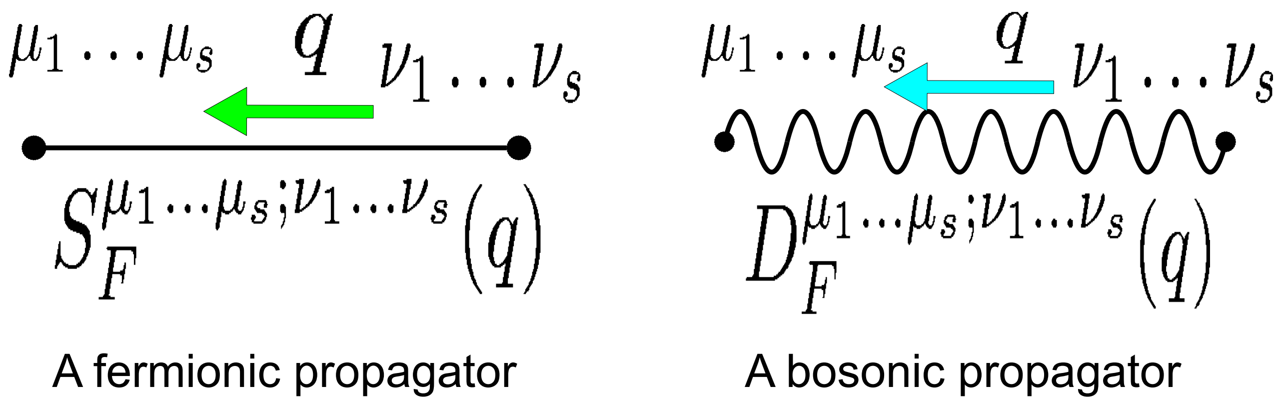


(c) An incidental antifermionic line



(d) A terminal antifermionic line

Figure 15: External fermionic or antifermionic lines for higher-spin fields



(a) A Feynman propagator for a spin-(s + 1/2) field

(b) A Feynman propagator for a spin-s field

Figure 16: Feynman propagators for higher-spin fields

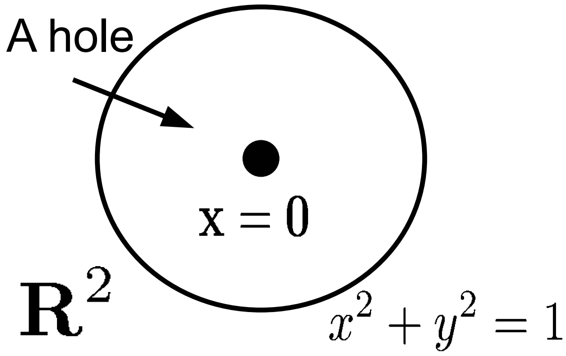
particle is related with $\nu_1 \dots \nu_s$ indices in these diagrams, while the terminal point involves $\mu_1 \dots \mu_s$ indices. There are many types of vertices, and some of them are shown in Figure 18 , where the hat symbol refers to the removal of the index on which the hat is put. In Figure 18a , $\gamma_{(\mu_1 \dots \hat{\mu}_k \dots \mu_s)}$ is a symmetrised γ matrix; here the it is defined as (3.4.20) . For example, we have

$$\gamma_{(\mu_1 \mu_2)} = \frac{1}{2}\left\{\gamma_{\mu_1}\gamma_{\mu_2} + \gamma_{\mu_2}\gamma_{\mu_1}\right\} \tag{8.1.1}$$

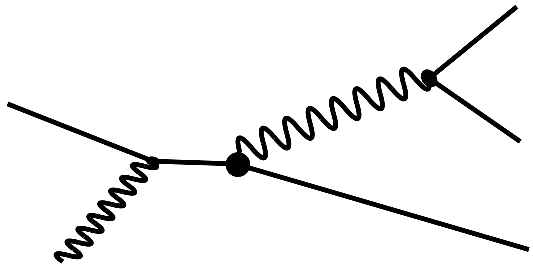
Similar rules hold true for $\gamma_{(\dots)}$ in Figure 18b . Let us emphasise that coupling constant g in Figure 18a is different from the coupling constant g in Figure 18b . External photons are shown in Figure 19 .

Now one employs another method instead of the perturbation method[45] for calculating scattering amplitudes. The new rule states that among all possible diagrams only the simply-connected diagrams in $\mathbf{R}^2$ are taken, and we calculate such simply-connected diagrams in $\mathbf{R}^2$ to obtain the scattering amplitudes. In addition, the new rule defined here states that loop diagrams have hole within the loop, and therefore it is not simply-connected in $\mathbf{R}^2$. Here by simply-connected we mean the object which can contract to a point smoothly. For example a circle $C : x^2 + y^2 = 1$ in $\mathbf{R} - \{\mathbf{0}\}$ cannot shrink to a point. See Figure 17a

[45]If we calculate scattering amplitudes by perturbation, the second-order terms must be much smaller than first-order ones as a general principle, but potentially the former contains divergent diagrams in many cases. This suggests that higher-order terms may have much more divergent terms, even though they must much smaller than the first-order terms. If we take renormalized perturbation theory, the theory may seem to be justified only when we conduct renomalization to remove such divergent terms before perturbation expansion.



(a) The new rule states that a loop has a hole within it. This prevents the diagram from being simply-connected in $\mathbf{R}^2$.



(b) An example for a simply-connected diagram in terms of a many-point function.

Figure 17: A loop has a hole within it, and so it cannot contract to a point smoothly in $\mathbf{R}^2$ (Fig 17a) . For a many-point function, we can construct a simply-connected diagram (Fig 17b).

. We have to mention that it is possible to make a simply-connected diagram for a many-point function such as Figure 17b . Let us emphasise that this new rule is not a perturbation method.

A decay of a charged massive spin-3/2 particle: As an example of the Feynman rule, let us consider a decay of a negatively charged massive spin-3/2 field. Assuming that the decay mode is

$$\psi^-_\mu(p,\lambda) \to e^-(p',\lambda') + \gamma(k,\lambda'') \quad , \tag{8.1.2}$$

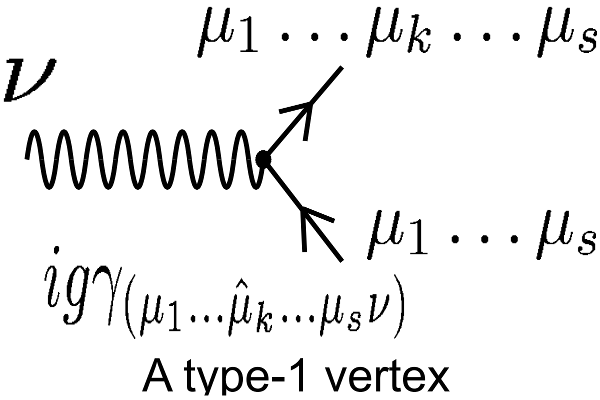
and that the spin-3/2 particle is at rest[46] initially, we draw the simply-connected Feynman diagram (See Figure 20a) in $\mathbf{R}^2$. Then we have

$$iM = \overline{u}_{p'\lambda'} ig\gamma_{(\mu\nu)} \epsilon^{*\nu}_{\lambda''}(k) U^\mu_\lambda(p) \quad , \tag{8.1.3}$$

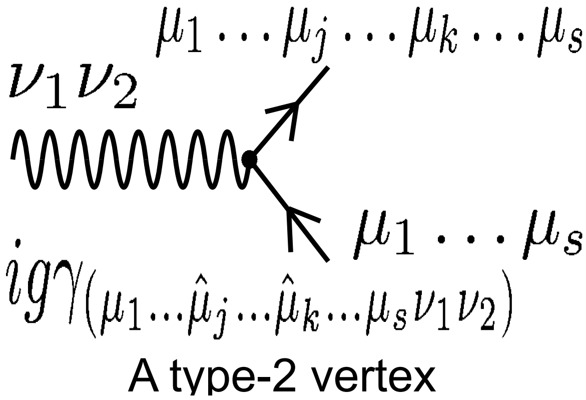
and the averaged scattering amplitude is

$$\frac{1}{24}\sum_{\lambda=-3/2}^{3/2}\sum_{\lambda'=-1/2}^{1/2}\sum_{\lambda''=-1}^{1}|iM|^2 = \frac{1}{24}\sum_{\lambda,\lambda',\lambda''}\left(\overline{u}_{p'\lambda'} ig\gamma_{(\mu\nu)}\epsilon^{*\nu}_{\lambda''}(k)U^\mu_\lambda(p)\right)\Big((-ig)\overline{U}^\alpha_\lambda\epsilon^\beta_{\lambda''}(k)\gamma_{(\alpha\beta)}u_{p'\lambda'}\Big) \quad . \tag{8.1.4}$$

[46]This means that $p = (E_{\mathbf{p}}, 0, 0, 0) = (m_{3/2}, \mathbf{0})$.

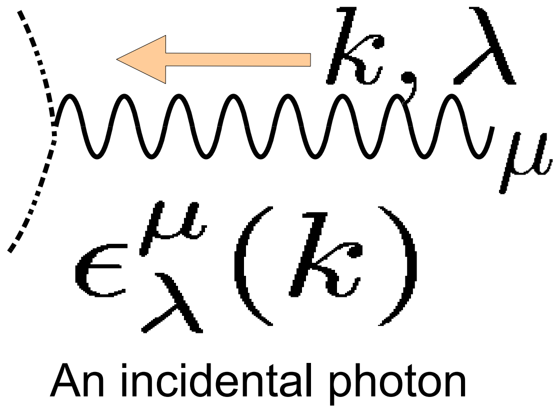


(a) A type-1 vertex in which the hat stands for the removal of its index

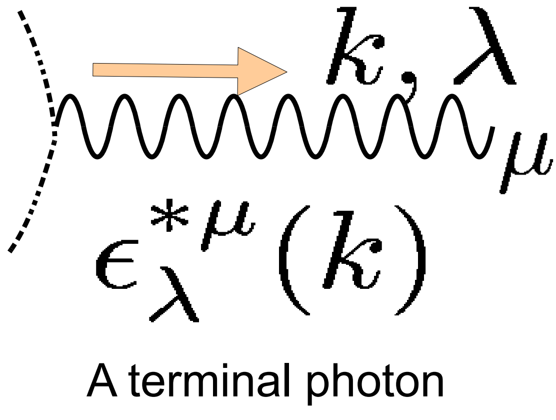


(b) A type-2 vertex in which the hats stand for the removal of their indices.

Figure 18: Several types of vertices are shown. Note that the coupling constant g in Figure 18a is different from that in Figure 18b .

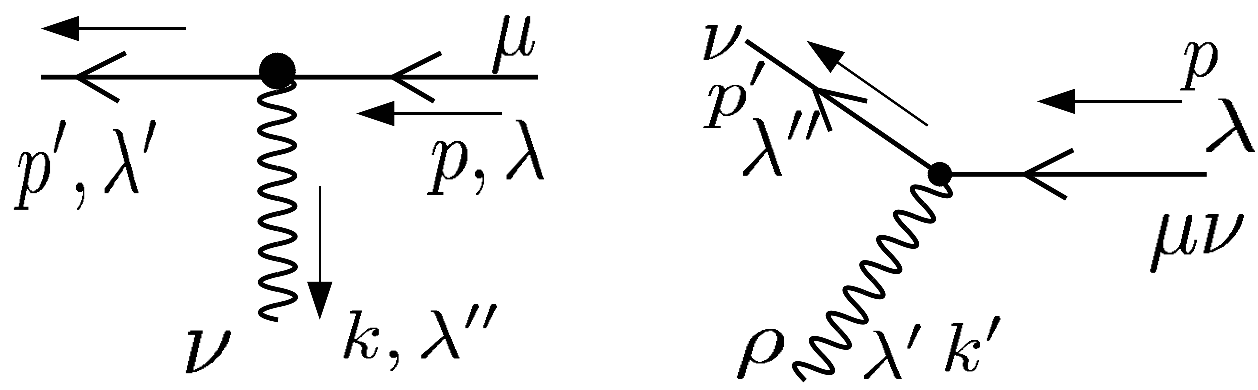


(a) An incidental photon



(b) A terminal photon

Figure 19: External photons



(a) In $\mathbf{R}^2$ the simply-connected Feynman diagram describing a decay of a charged massive spin-3/2 particle

(b) In $\mathbf{R}^2$ the simply-connected Feynman diagram for a decay of a charged massive spin-5/2 particle

Figure 20: In $\mathbf{R}^2$ the simply-connected Feynman diagrams for decays of particles

With the aid of (6.1.11) and gamma matrix formulae[47] , the amplitude becomes

$$\frac{1}{24}\sum_{\lambda,\lambda',\lambda''}|iM|^2 = \frac{1}{3}g^2(p\cdot p' + m_{3/2}^2) = \frac{g^2}{3}\left(E_{\mathbf{p}}E_{\mathbf{p'}} + m_{3/2}^2\right) \tag{8.1.5}$$

where $m_{3/2}$ is the mass of the spin-3/2 particle. In this case the differential decay rate is

$$d\Gamma = \frac{1}{2m_{3/2}}\left(\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p'}}}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}}}\right)\times\left(\frac{1}{24}\sum_{\lambda,\lambda',\lambda''}|iM|^2\right)(2\pi)^4\delta^{(4)}(k+p'-p) \quad , \tag{8.1.6}$$

and so after some[48] calculation, the decay rate

$$\Gamma = \frac{1}{96\pi}g^2 m_{3/2}\left(1-\frac{m_e^2}{m_{3/2}^2}\right)\left(3+\frac{m_e^2}{m_{3/2}^2}\right) \tag{8.1.7}$$

is derived.

[47]See Appendix Formulae for gamma matrices.

[48]Here the formula $\delta(F(x)) = \frac{1}{F'(x_0)}\delta(x-x_0)$, $\quad (F(x_0)=0)$ is used.

8.2 A decay of a massive spin-5/2 particle

Now let us consider a decay of charged massive particle with spin-5/2 . Assuming that its decay mode is

$$\psi_{\mu\nu}(p,\lambda) \to \psi^-_\mu(p',\lambda'') + \gamma(k',\lambda') \quad , \tag{8.2.1}$$

and so its Feynman diagram is written in Figure 20b . Accordingly the scattering matrix is

$$iM = \overline{U}^{\xi}_{\lambda''}(p') ig\gamma_{(\mu\nu\xi\rho)}\epsilon^{*\rho}_{\lambda'}(k')U^{\mu\nu}_{\lambda}(p) \tag{8.2.2}$$

and its complex conjugate is

$$(iM)^* = \overline{U}^{\mu\nu}_{\lambda}(p)\epsilon^{\rho}_{\lambda'}(k')\gamma_{(\mu\nu\xi\rho)}(-ig)U^{\xi}_{\lambda''}(p') \quad . \tag{8.2.3}$$

After relatively tough calculation, we have the averaged scattering amplitude:

$$\frac{1}{72}\sum_{\lambda,\lambda',\lambda''}|iM|^2 = g^2(p'\cdot p + m_{3/2}m_{5/2}). \tag{8.2.4}$$

In deriving (8.2.4) , we use the formula (6.1.58) , that is,

$$\sum_{\lambda=-5/2}^{5/2} U^{\mu\nu}_{\lambda}(p)\overline{U}^{\alpha\beta}_{\lambda}(p) = \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\not p + m_{5/2}\right) \quad , \tag{8.2.5}$$

and gamma matrix formulae (C.1.7) (See Appendix Formulae for gamma matrices) .

Provided[49] that the initially the spin-5/2 particle is at rest, that is, $p = (m_{5/2}, \mathbf{0})$, we calculate the differential decay rate:

$$d\Gamma = \frac{1}{2m_{5/2}}\left(\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}'}}\right)\times\left(\frac{1}{72}\sum|iM|^2\right)\times(2\pi)^4\delta^{(4)}(k'+p'-p) \quad , \tag{8.2.6}$$

where $m_{5/2}$, $m_{3/2}$ are the masses of the spin-5/2 and the spin-3/2 particles, respectively. With the aid of

$$\delta(F(\tau)) = \frac{1}{F'(t')}\delta(\tau - t') \quad , \tag{8.2.7}$$

the differential decay rate becomes

$$d\Gamma = \frac{2\pi}{2m_{5/2}}\left(\frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\frac{1}{|\mathbf{p}'|}\right)\left(\frac{1}{72}\sum|iM|^2\right)\times\frac{m^2_{5/2}+m^2_{3/2}}{2m^2_{5/2}}\times\delta(|\mathbf{p}'|-|\mathbf{p}'_0|) \quad , \tag{8.2.8}$$

[49]This assumption may plausible as long as the spin-5/2 particle is heavy like weak bosons.

where $|\mathbf{p'}_0| = \dfrac{m_{5/2}^2 - m_{3/2}^2}{2m_{5/2}}$. Consequently the decay rate of this spin-5/2 particle

$$\Gamma = \frac{g^2}{32\pi} m_{5/2} \left\{ 1 - \left(\frac{m_{3/2}}{m_{5/2}} \right)^2 \right\} \left(1 + \frac{m_{3/2}}{m_{5/2}} \right)^2 \tag{8.2.9}$$

is derived.

CONCLUSION

In this dissertation, higher-spin fields were discussed. The equations of motion for massless higher-spin fields were established, and the number of degrees of freedom for these fields were counted. It turned out that Rarita-Schwinger fields coupling with an external electromagnetic field behaved in an acausal way, and the causality was restored by supergravity at a classical level. A quantisation for higher-spin fields was conducted with the aid of generalised polarisation 4-vectors. Accordingly the Feynman rule for higher-spin fields was constructed in momentum space.

One of remaining problems is whether or not we can apply, to existing fields, these results such as the equations of motion. A scalar meson is a composite particle of quarks, and is described by Klein-Gordon equations. Similarly it seems to reasonable to think that a composite higher-spin particle is described corresponding equations of motion.

Statistical mechanics for higher-spin fermions are also considered. Basically grandcanonical ensemble for fermionic particles are designed for mostly Dirac particles. It is expected that a statistical formalism for the ensemble of higher-spin fermions will be established soon.

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APPENDICES

A Dimensional analysis for field theories

In this section, a brief review of dimensional analysis for field theories is done, especially for QED and linearlised gravity. First we consider a 4-dimensional Lagrangian density in four dimension for classical electrodynamics:

$$\mathcal{L}_{CED} = \frac{-1}{16\pi c^2} F^{\mu\nu} F_{\mu\nu} + \frac{-1}{c^2} A_\mu j^\mu \quad \left(\frac{\mathrm{erg} \cdot \mathrm{sec}}{\mathrm{cm}^4}\right) \quad , \tag{A.0.10}$$

where $j^\mu = \rho(c, \mathbf{v})$ is called the current four-vector[50], and

$$j^\mu = \rho \frac{dx^\mu}{dt} = \sum_a e_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a) \frac{dx^\mu}{dt} \tag{A.0.11}$$

is its definition. The reason why the Lagrangian density takes such a dimension is that the action $S = \int d^4x \mathcal{L}_{\mathrm{CED}}$ has the dimension of angular momentum.

A.1 QED Lagrangian

Second let us consider a Lagrangian density for quantum electrodynamics. Here, natural unit system $\hbar = c = 1$ is adopted. In this unit system, any physical dimension can be expressed with power of length; we may define the length dimension by $1 = [\mathrm{length}] = [\mathrm{time}] = [\mathrm{mass}^{-1}] = [\mathrm{momentum}^{-1}] = [\mathrm{energy}^{-1}]$, where we express the length dimension of a quantity Q as $[Q]$. In this case, the action

$$S = \int d^4x \mathcal{L}_{\mathrm{QED}} \tag{A.1.1}$$

is dimensionless[51], and accordingly the Lagrangian density has the dimension of length^{-4} , that is,

$$[\mathcal{L}_{\mathrm{QED}}] = -4 \quad , \tag{A.1.2}$$

[50] Reasonably, this is a four-vector. To see it, starting by the identity that $de = \rho dV$, we multiply both sides by dx^μ . It follows that $dedx^\mu = \rho dV dt \frac{dx^\mu}{dt}$, and since in this equation the left hand side is a four-vector(due to de being a scalar), the right hand side must be so. The quantity $dVdt$ is a Lorentz scalar, and so $\rho \frac{dx^\mu}{dt}$ must be a four-vector.

[51] In this unit system, angular momentum is dimensionless.

In a general d dimension,

$$[\mathcal{L}_{\mathrm{QED}}] = -d \tag{A.1.3}$$

is obtained. Specifically, the QED Lagrangian density is

$$\mathcal{L}_{\mathrm{QED}} = \overline{\psi}\left\{i\gamma^{\mu}(\partial_{\mu} - iqA_{\mu}) - m\right\}\psi + \frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu} \tag{A.1.4}$$

$$= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + qA_{\mu}\overline{\psi}\gamma^{\mu}\psi + \frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu} \tag{A.1.5}$$

$$= \mathcal{L}_{\mathrm{Dirac}} + \mathcal{L}_{\mathrm{EM}} \quad , \tag{A.1.6}$$

where q is the charge[52] of the electric source, and we define the particle current density four-vector as $j^{\mu} = \overline{\psi}\gamma^{\mu}\psi$. It follows that the EM Lagrangian density becomes

$$\mathcal{L}_{\mathrm{EM}} = qA_{\mu}j^{\mu} + \frac{-1}{16\pi}F^{\mu\nu}F_{\mu\nu} \quad . \tag{A.1.7}$$

Now we study the dimensions of fields and coupling constants. Equations are

$$2[\psi] - 1 = -d \quad , \quad 2[\psi] + [m] = -d \tag{A.1.8}$$

$$2[A_{\mu}] - 2 = -d \quad , \quad [e] + [A_{\mu}] + 2[\psi] = -d \quad , \tag{A.1.9}$$

and it follows that

$$[\psi] = \frac{1-d}{2}, \ \ [m] = -1, \ \ [A_{\mu}] = \frac{2-d}{2}, \ \ [e] = \frac{d-4}{2} \quad , \tag{A.1.10}$$

where that the dimension of mass is [length^{-1}] is consistent with its definition. The fact that electron is dimensionless in 4-dimensional case is drawn attention to. In the above, we find that this fact is related with the renormalizability of QED.

A.2 Lagrangian for linearised gravity

For the system of a weak gravitational field, its metric tensor can be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \tag{A.2.1}$$

[52] $q = C \times e$, where $|e|$ is the elementary charge and C is a constant.

and for the purpose of keeping[53] its consistency, we have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad , \tag{A.2.2}$$

where $h_{\mu\nu}$ substantially may play a role in weak gravitation. Since the field is weak, the quadradic form of it and related similar objects[54] should vanish. Immediately the (linearlised) Christoffel symbol

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\mu}\left\{\partial_\beta h_{\gamma\mu} + \partial_\gamma h_{\mu\beta} - \partial_\mu h_{\beta\gamma}\right\} \tag{A.2.3}$$

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left\{\partial_\beta h_{\gamma\alpha} + \partial_\gamma h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\right\} \quad , \tag{A.2.4}$$

and after some calculation we can show that

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\gamma\beta\alpha} = \frac{1}{4}\left\{2(\partial^\gamma h^{\alpha\beta})(\partial_\alpha h_{\beta\gamma}) - (\partial^\gamma h^{\alpha\beta})(\partial_\gamma h_{\beta\alpha})\right\} \tag{A.2.5}$$

$$= \Gamma^{\alpha\beta\gamma}\Gamma_{\beta\gamma\alpha} \quad . \tag{A.2.6}$$

Further, the relation between the linearised Christoffel symbol and rank-1 linear Christoffel symbol for spin-2 is

$$\Gamma^{(1)}_{\beta;\mu_1\mu_2} = -2\Gamma_{\beta\mu_1\mu_2} \quad . \tag{A.2.7}$$

Now the (linearised) Riemann curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2}\eta^{\alpha\epsilon}\left\{\partial_\mu\partial_\beta h_{\nu\epsilon} - \partial_\nu\partial_\beta h_{\mu\epsilon} + \partial_\nu\partial_\epsilon h_{\beta\mu} - \partial_\mu\partial_\epsilon h_{\beta\nu}\right\} + (\text{higher order terms}) \tag{A.2.8}$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left\{\partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\nu\partial_\beta h_{\mu\alpha} + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\mu\partial_\alpha h_{\nu\beta}\right\} + (\text{higher order terms}) \tag{A.2.9}$$

are derived. Similarly, the relation between the linearised Riemann curvature tensor and the rank-2 generalised linear Riemann curvature tensor for spin-2 $R_{\beta_1\beta_2;\mu_1\mu_2}$

$$R_{\beta_1\beta_2;\mu_1\mu_2} \equiv \frac{-1}{2}\left(R_{\beta_1\mu_1\beta_2\mu_2} + R_{\beta_1\mu_2\beta_2\mu_1}\right) \tag{A.2.10}$$

$$= \partial_{\beta_1}\partial_{\beta_2}h_{\mu_1\mu_2} + \partial_{\mu_1}\partial_{\mu_2}h_{\beta_1\beta_2} + \frac{-1}{2}\left\{\partial_{\beta_1}\partial_{\mu_1}h_{\beta_2\mu_2} + \partial_{\beta_1}\partial_{\mu_2}h_{\beta_2\mu_1} + \partial_{\mu_1}\partial_{\beta_2}h_{\beta_1\mu_2} + \partial_{\beta_2}\partial_{\mu_2}h_{\beta_1\mu_1}\right\}$$
$$+ (\text{higher order terms}) \tag{A.2.11}$$

[53] If we take the tensor as $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, then $h_{\mu\nu}$ becomes antisymmetric.

[54] For example, $h_{\mu\nu}\partial_\zeta h^{\zeta\kappa} = 0$.

is derived. The (linearlised) Ricci tensor is

$$R_{\beta\nu} = \frac{1}{2}\left\{\partial^{\epsilon}\partial_{\beta}h_{\nu\epsilon} - \partial_{\nu}\partial_{\beta}h^{\alpha}_{\ \alpha} + \partial^{\alpha}\partial_{\nu}h_{\beta\alpha} - \partial_{\alpha}\partial^{\alpha}h_{\nu\beta}\right\} + (\text{higher order terms}) \quad , \tag{A.2.12}$$

and the (linearised) Ricci scalar is

$$R = R^{\alpha}_{\ \alpha} = \partial^{\epsilon}\partial^{\nu}h_{\nu\epsilon} - \partial_{\epsilon}\partial^{\epsilon}h^{\mu}_{\ \mu} + (\text{higher order terms}) \quad . \tag{A.2.13}$$

Now we choose gauge conditions[55] as

$$\partial^{\alpha}\partial^{\beta}h_{\alpha\beta} = 0 \ , \quad \partial^{\alpha}\partial_{\alpha}h^{\beta}_{\ \beta} = 0 \quad , \tag{A.2.14}$$

and in these conditions the lowest-order terms of the Ricci scalar vanishes. In this gauge the Ricci scalar becomes

$$R = -\Gamma^{\mu\nu\xi}\Gamma_{\xi\nu\mu} \quad . \tag{A.2.15}$$

Accordingly the linearlised Einstein equation, where only the lowest order of $h_{\mu\nu}$ is taken, is expressed as

$$-\left\{\partial^{\alpha}\partial_{\alpha}h_{\mu\nu} - \partial^{\epsilon}(\partial_{\mu}h_{\nu\epsilon} + \partial_{\nu}h_{\mu\epsilon}) + \partial_{\mu}\partial_{\nu}h^{\epsilon}_{\ \epsilon}\right\} = 0 \tag{A.2.16}$$

in vacuum. The action of the linearlised gravitational field is

$$S_g = \frac{-c^3}{16\pi G}\int d^4x R\sqrt{-g} = \frac{-c^3}{16\pi G}\int d^4x(-\Gamma^{\mu\nu\xi}\Gamma_{\xi\nu\mu}) \quad , \tag{A.2.17}$$

where $G = 6.67 \times 10^{-8}\ \frac{\text{esu}^2}{\text{g}^2}$ is the gravitational constant.

Quantum linearised gravity: In quantum mechanics, the action for the linearised gravity in 4-dimensional space is

$$S_g = \frac{-1}{16\pi G}\int d^4x\left(-\Gamma^{\mu\nu\xi}\Gamma_{\xi\nu\mu}\right) \quad , \tag{A.2.18}$$

and in this case $[G] = 2$ and $[S_g]$ is dimensionless. Accordingly we have

$$2\left(2[h_{\mu\nu}] - 1\right) + 4 - 2 = 0 \tag{A.2.19}$$

$$\text{and so} \quad [h_{\mu\nu}] = 0 \tag{A.2.20}$$

[55]Here the de Donger gauge is taken: $\partial^{\lambda}h_{\lambda\mu} - \frac{1}{2}\partial_{\mu}h^{\lambda}_{\ \lambda} = 0$. In addition we adopt $h^{\lambda}_{\ \lambda} = 0$, and so $\partial^{\lambda}h_{\lambda\mu} = 0$ is derived.

is derived. In a general d-dimensional case,

$$S_g = \frac{-1}{16\pi G} \int d^d x \left(-\Gamma^{\mu\nu\xi} \Gamma_{\xi\nu\mu} \right) \quad , \tag{A.2.21}$$

and so

$$2\left(2[h_{\mu\nu}] - 1\right) + d - 2 = 0 \tag{A.2.22}$$

$$\text{that is,} \quad [h_{\mu\nu}] = \frac{4-d}{4} \tag{A.2.23}$$

is obtained.

B QED results

Here we show some results in terms of QED. These results are based on Berestetskii, Lifshitz and Pitaevskii[15] .

B.1 Bremsstrahlung electron-nucleus

In quantum electrodynamics, radiation is an important issue. As we know in classical electrodynamics, the effective radiation

$$\xi = \int_0^{+\infty} \Delta \cdot 2\pi\rho d\rho \tag{B.1.1}$$

has the dimension of energy times area. The ratio of ξ to the energy of the radiating system is called the cross section for energy loss by radiation.

Bremsstrahlung (braking radiation) is a phenomenon in which a charged particle emits radiation in a collision with other charged particles.

Electron-nucleaus bremsstrahlung: The differential photon-emission cross section for electron-nucleus bremsstrahlung is

$$d\sigma = \frac{1}{(2\pi)^5} |iM|^2 \frac{\omega|\mathbf{p}'|}{|\mathbf{p}|} d\Omega_{\mathbf{k}} d\Omega' d\omega \tag{B.1.2}$$

$$= \frac{Z^2 \alpha r_e^2}{4\pi^2} \frac{p' m^4}{p q^4} \frac{d\omega}{\omega} d\Omega_{\mathbf{k}} d\Omega' \times \left\{ \frac{q^2}{\kappa\kappa' m^2} (2\epsilon^2 + 2\epsilon'^2 - q^2) \right.$$

$$\left. + q^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa'} \right)^2 - 4 \left(\frac{\epsilon}{\kappa'} - \frac{\epsilon'}{\kappa} \right)^2 + \frac{2\omega q^2}{m^2} \left(\frac{1}{\kappa'} - \frac{1}{\kappa} \right) - \frac{2\omega^2}{m^2} \left(\frac{\kappa'}{\kappa} + \frac{\kappa}{\kappa'} \right) \right\} \tag{B.1.3}$$

where $d\Omega_{\mathbf{k}}$ stands for the differential solid angle for the momentum $\mathbf{k}$ of a photon. In addition,

$$r_e = \frac{e^2}{m} \ , \quad \kappa = \epsilon - \mathbf{n}\cdot\mathbf{p} \ , \quad \kappa' = \epsilon' - \mathbf{n}\cdot\mathbf{p}' \tag{B.1.4}$$

$$\mathbf{n} = \frac{\mathbf{k}}{\omega} \ , \quad \mathbf{q} = \mathbf{p}' + \mathbf{k} - \mathbf{p} \quad . \tag{B.1.5}$$

In the non-relativistic case, the bremsstrahlung cross-section becomes

$$d\sigma_\omega = \frac{16}{3\hbar c}\left(\frac{e_1 e_2}{c}\right)^2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \frac{1}{v^2}\frac{d\omega}{\omega} \ln\left(\frac{v+v'}{v-v'}\right) \quad . \tag{B.1.6}$$

B.2 Basic results for QED

This subsection shows some basic results for QED, and so those who are familiar with these results may skip this subsection.

Helicity for plane wave solutions of Dirac equations: Solving the Dirac equations, we have plane wave solutions:

$$\psi_p = \frac{1}{\sqrt{\epsilon}} u_p e^{-ip\cdot x} \quad , \quad \psi_{-p} = \frac{1}{\sqrt{\epsilon}} v_p e^{ip\cdot x} \tag{B.2.1}$$

for a particle and its antiparticle, respectively, where the energy $\epsilon = +\sqrt{\mathbf{p}^2 + m^2}$ of the particle is taken to be the positive quantity. Specifically, in the standard representation[56]

$$u_{p\lambda} = \begin{pmatrix} \sqrt{\epsilon + m}\,\omega_\lambda \\ \sqrt{\epsilon - m}(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\sigma})\omega_\lambda \end{pmatrix} \quad , \quad v_{p\lambda'} = \begin{pmatrix} \sqrt{\epsilon - m}(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\sigma})\omega'_\lambda \\ \sqrt{\epsilon + m}\,\omega'_\lambda \end{pmatrix} \tag{B.2.2}$$

$$\lambda = -\frac{1}{2}, \frac{1}{2} \tag{B.2.3}$$

where λ is the helicity[57] of the particle or antiparticle. In addition, ω_λ is a two-component quantity which is linked to the helicity states of the particle, and it is the eigenfunction of the operator $(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\sigma})$:

$$\frac{1}{2}(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\sigma})\omega_\lambda = \lambda\omega_\lambda \quad , \tag{B.2.4}$$

[56]In this representation, $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[57]The helicity of a particle is the component of its spin in the direction of its momentum $\mathbf{p}$.

and one can express the eigenfunctions as

$$\omega_{\frac{1}{2}} = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin\frac{\theta}{2} \end{pmatrix} \quad , \quad \omega_{-\frac{1}{2}} = \begin{pmatrix} -e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \cos\frac{\theta}{2} \end{pmatrix} \quad , \tag{B.2.5}$$

where θ and ϕ stand for, respectively, the polar angle and the azimuth of the direction of $\frac{\mathbf{p}}{|\mathbf{p}|}$ relative to fixed axis xyz. One can see

$$\omega^{\dagger}_{\lambda'}\omega_{\lambda} = \delta_{\lambda',\lambda} \tag{B.2.6}$$

by simple calculations. On the other hand, ω'_{λ} is the corresponding eigenfunction of that operator for the antiparticle. It is taken as

$$\omega'_{\lambda} = -\sigma_y\omega_{-\lambda} = (-1)^{\lambda-\frac{1}{2}} i\omega^{*} \quad , \tag{B.2.7}$$

and it follows that

$$\omega'^{\dagger}_{\lambda'}\omega'_{\lambda} = \delta_{-\lambda',-\lambda} \tag{B.2.8}$$

$$\frac{1}{2}(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\boldsymbol{\sigma})\omega'_{\lambda} \neq \lambda\omega'_{\lambda} \quad . \tag{B.2.9}$$

from (B.2.5) and (B.2.6) . With the aid of (B.2.4) , one can write $u_{p\lambda}$, $v_{p\lambda'}$ as

$$u_{p\frac{1}{2}} = \begin{pmatrix} \sqrt{\epsilon+m}\,\omega_{\frac{1}{2}} \\ \sqrt{\epsilon-m}\,\omega_{\frac{1}{2}} \end{pmatrix} \quad , \quad u_{p-\frac{1}{2}} = \begin{pmatrix} \sqrt{\epsilon+m}\,\omega_{-\frac{1}{2}} \\ -\sqrt{\epsilon-m}\,\omega_{-\frac{1}{2}} \end{pmatrix} \tag{B.2.10}$$

and accordingly

$$\overline{u}_{p\lambda'}u_{p\lambda} = 2m\delta_{\lambda',\lambda} \tag{B.2.11}$$

$$\lambda, \lambda' = -\frac{1}{2}, \frac{1}{2} \tag{B.2.12}$$

is derived. We also have

$$\overline{v}_{p\lambda'}v_{p\lambda} = -2m\delta_{-\lambda',-\lambda} \tag{B.2.13}$$

by using (B.2.8) . Here, in terms of the eigenfunctions, let us show important formulae:

$$\omega_{\frac{1}{2}}\omega^{\dagger}_{\frac{1}{2}} + \omega_{-\frac{1}{2}}\omega^{\dagger}_{-\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.2.14}$$

$$\omega_{\frac{1}{2}}\omega^{\dagger}_{\frac{1}{2}} - \omega_{-\frac{1}{2}}\omega^{\dagger}_{-\frac{1}{2}} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} = \boldsymbol{\sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|} \quad , \tag{B.2.15}$$

where we can use

$$\frac{\mathbf{p}}{|\mathbf{p}|} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \quad , \tag{B.2.16}$$

and

$$\omega'_{\frac{1}{2}}\omega'^{\dagger}_{\frac{1}{2}} + \omega'_{-\frac{1}{2}}\omega'^{\dagger}_{-\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \omega'_{\frac{1}{2}}\omega'^{\dagger}_{\frac{1}{2}} - \omega'_{-\frac{1}{2}}\omega'^{\dagger}_{-\frac{1}{2}} = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{pmatrix} \quad . \tag{B.2.17}$$

Helicity for vector particles: Solving Proca equations, we have the plane wave solutions

$$\phi^\mu = \frac{1}{\sqrt{2\epsilon}}\epsilon^\mu e^{-p\cdot x} \tag{B.2.18}$$

$$\epsilon_\mu \epsilon^{\mu *} = -1 \quad , \quad u_\mu p^\mu = 0 \tag{B.2.19}$$

where ϵ^μ is the unit polarisation 4-vector[58]. The solutions are expanded with these plane waves

$$\phi^\mu = \sum_{\mathbf{p},\lambda} \frac{1}{\sqrt{2\epsilon}} \left\{ A_{\mathbf{p}}\epsilon^\mu e^{-ip\cdot x} + B^\dagger_{\mathbf{p}}\epsilon^{*\mu} e^{ip\cdot x} \right\} \tag{B.2.20}$$

C Formulae for gamma matrices

This section shows some formulae for gamma matrices.

$$\gamma_\alpha \not{k} \gamma^\alpha = -2\not{k} \ , \quad \gamma_\alpha \not{k}\not{p}\gamma^\alpha = 4(k\cdot p) \tag{C.0.21}$$

$$tr\mathbf{1} = 4 \ , \quad tr(\not{p}\not{k}) = 4(p\cdot k) \tag{C.0.22}$$

$$tr(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_{2m+1}}) = 0 \tag{C.0.23}$$

$$tr(\not{k}_1\not{k}_2\not{k}_3\not{k}_4) = 4\left\{(k_1\cdot k_2)(k_3\cdot k_4) - (k_1\cdot k_3)(k_2\cdot k_4) + (k_1\cdot k_4)(k_3\cdot k_2)\right\} \tag{C.0.24}$$

In terms of γ^5 , we have

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\xi\rho}\gamma^\mu\gamma^\nu\gamma^\xi\gamma^\rho \tag{C.0.25}$$

$$\gamma^5\gamma^c = \frac{i}{3!}\epsilon_{\mu\nu\xi\rho}\gamma^\mu\gamma^\nu\gamma^\xi\eta^{\rho c} \quad . \tag{C.0.26}$$

[58] In (B.2.18) , ϵ in the denominator is the energy of the particle.

Proof of (C.0.25) is obvious from the anti-commutativity of gamma matrices. For (C.0.26) we can show that

$$\frac{4!}{i}\gamma^5\gamma^c = \epsilon_{\mu\nu\xi\rho}\gamma^\mu\gamma^\nu\gamma^\xi\left(2\eta^{\rho c} - \gamma^c\gamma^\rho\right) \tag{C.0.27}$$

$$= \epsilon_{\mu\nu\xi\rho}4\eta^{\rho c}\gamma^\mu\gamma^\nu\gamma^\xi + \epsilon_{\mu\nu\xi\rho}\gamma^\mu\gamma^\nu\gamma^c\gamma^\xi\gamma^\rho \tag{C.0.28}$$

$$= \epsilon_{\mu\nu\xi\rho}8\eta^{\rho c}\gamma^\mu\gamma^\nu\gamma^\xi + \gamma^c\epsilon_{\mu\nu\xi\rho}\gamma^\mu\gamma^\nu\gamma^\xi\gamma^\rho \tag{C.0.29}$$

$$= \epsilon_{\mu\nu\xi\rho}8\eta^{\rho c}\gamma^\mu\gamma^\nu\gamma^\xi + \gamma^c\frac{4!}{i}\gamma^5 \quad , \tag{C.0.30}$$

and taking advantage of the anti-commutativity of γ^5 we obtain (C.0.26) .

C.1 Symmetrised gamma matrices

As previously stated, the symmetrised gamma matrix is

$$\gamma^{(\mu_1\mu_2\ldots\mu_m)} = \frac{1}{m!}\{\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_m} + \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_m}\gamma^{\mu_{m-1}} \tag{C.1.1}$$

$$+\cdots+\gamma^{\mu_m}\gamma^{\mu_{m-1}}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\} \quad . \tag{C.1.2}$$

Let us now write the matrices explicitly:

$$\gamma^{(\mu\nu)} = \eta^{\mu\nu} \tag{C.1.3}$$

$$\gamma^{(\alpha\beta\gamma)} = \frac{1}{3}\left\{\gamma^\alpha\eta^{\beta\gamma} + \gamma^\beta\eta^{\alpha\gamma} + \gamma^\gamma\eta^{\beta\alpha}\right\} \tag{C.1.4}$$

$$\gamma^{(\mu\nu\xi\rho)} = \eta^{\mu\nu}\eta^{\xi\rho} + \eta^{\mu\xi}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\xi\nu} \tag{C.1.5}$$

$$\gamma_{(\alpha\beta)}\gamma^{(\alpha\beta)} = 4 \tag{C.1.6}$$

$$\gamma_{(\mu\nu\xi\rho)}\gamma^{(\mu\nu\xi\rho)} = 72 \tag{C.1.7}$$

C.2 Antisymmetrised gamma matrices:

Antisymmetrised gamma matrices are given as

$$\gamma^{[\mu_1\mu_2\ldots\mu_s]} = \frac{1}{s!}\sum \mathrm{sgn}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_s} \quad , \tag{C.2.1}$$

and specifically we have

$$\gamma^{[\alpha\beta]} = \frac{1}{2!}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right) = \eta^{\alpha\beta} - \gamma^\beta\gamma^\alpha = -\eta^{\alpha\beta} + \gamma^{\alpha\beta} \tag{C.2.2}$$

$$\gamma^{[\alpha\beta\gamma]} = -\gamma^\alpha\eta^{\beta\gamma} + \gamma^\beta\eta^{\alpha\gamma} - \gamma^\gamma\eta^{\beta\alpha} + \gamma^\alpha\gamma^\beta\gamma^\gamma \tag{C.2.3}$$

and so on. Related formulae are written as

$$\gamma_\beta\gamma^{[\beta\gamma]} = 3\gamma^\gamma \tag{C.2.4}$$

$$\gamma_\beta\gamma^{[\alpha\beta\gamma]} = 2\eta^{\gamma\alpha} - 2\gamma^\alpha\gamma^\gamma \tag{C.2.5}$$

$$\gamma^{[\mu\nu\xi]}D_\mu = -i\gamma^5\epsilon^{\nu\alpha\beta\xi}\gamma_\alpha D_\beta\ , \quad \epsilon^{0123} = -1 = -\epsilon_{0123} \tag{C.2.6}$$

Note: This project for the master's dissertation started on 11 June 2011 and finished on 26 September 2011.